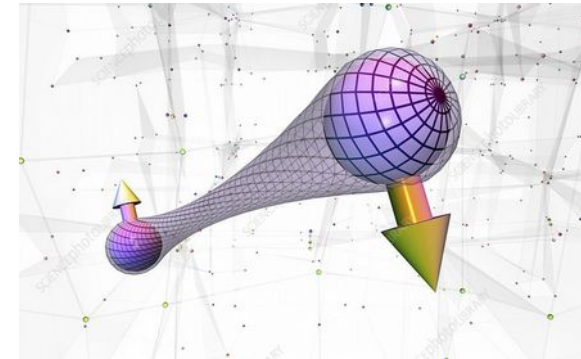


Quantum tomography of diboson systems at colliders: entanglement, Bell violations and new physics



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Foundational tests of quantum mechanics at the LHC

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in collaboration with: Marco Fabbrichesi, Roberto Floreanini, and Luca Marzola

based on: **arXiv: 2302.00683 [hep-ph]**

Outline

- Polarization **density** matrix of **diboson** production: theoretical tools and analytical results
- Quantum tomography, **entanglement** and **Bell violation observables**:
 - ▶ Higgs decays $H \rightarrow WW^*, ZZ^*$ at the LHC
 - ▶ Drell-Yan processes: $pp \rightarrow WW, ZZ, WZ$ at the **LHC** and **future lepton colliders**
 - ▶ **Bounds on** new anomalous **HWW** and **HZZ** couplings using quantum tomography
- Conclusions

Quantum tomography of two Vector-Boson production

- Requires the knowledge of the **polarization density matrix** for two vector bosons (WW, ZZ, WZ)
- it can be fully reconstructed from the angular distributions of the **VB decay products**
- **but it can also be computed analytically** (see next slides)
- so far experimental analyses have been focused on the density matrix of two spin $\frac{1}{2}$ → top-quark
- no measurement so far at LHC of the density matrix for Di-Boson production
- knowledge of the full **polarization density matrix** allows to study many interesting phenomena

- ▶ Quantum Entanglement
- ▶ Violation of Bell's inequalities
- ▶ Sensitivity to New Physics

Polarization vectors of spin-1 particle

choose a

right-handed basis

$$\{\hat{\mathbf{n}}, \hat{\mathbf{r}}, \hat{\mathbf{k}}\}$$

$$\hat{\mathbf{n}} = \hat{\mathbf{r}} \times \hat{\mathbf{k}}$$

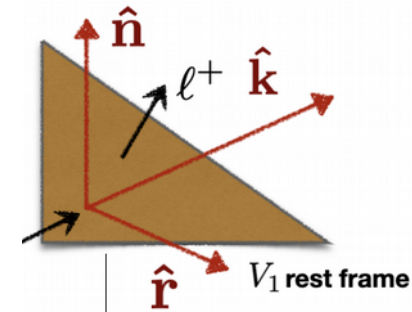
Spin-1 eigenstates

● on rest frame

$$p_\mu = (m, 0)$$

$$\psi_\pm = -\frac{1}{\sqrt{2}} (\pm \hat{\mathbf{n}} + i \hat{\mathbf{r}}) \quad \psi_0 = \hat{\mathbf{k}}$$

corresponding to eigenvalues $\lambda = \pm 1, 0$



decay plane of V_1 at rest

direction of spin-axis quantization

(covariant formalism)

● In a more general frame

(performing a Lorentz boost along $\hat{\mathbf{k}}$)

$$p^\mu = E(1, \hat{\mathbf{k}}\beta)$$

velocity

particle energy

boosted basis

$$\{\hat{\mathbf{n}}, \hat{\mathbf{r}}, \hat{\mathbf{k}}\}$$



$$n_1^\mu = (0, \hat{\mathbf{n}}), \quad n_2^\mu = (0, \hat{\mathbf{r}}), \quad n_3^\mu = \frac{E}{M}(\beta, \hat{\mathbf{k}})$$

$$n_0^\mu = E/M(1, \hat{\mathbf{k}}\beta)$$



$$g_{\mu\nu} n_m^\mu n_n^\nu = -\delta_{mn}$$

$\{n, m\} = 0, 1, 2, 3$

orthogonal to the particle 4-momentum

$$n_m^\mu p_\mu = 0$$

covariant polarization vector of spin-1

rest frame limit

$$\varepsilon^\mu(p, \lambda) = -\frac{1}{\sqrt{2}}|\lambda| (\lambda n_1^\mu + i n_2^\mu) + (1 - |\lambda|) n_3^\mu$$

$$\varepsilon^\mu(p, \lambda) \xrightarrow{(\beta \rightarrow 0)} \psi_\pm, \psi_0$$

helicity
 $\lambda = \pm 1, 0$

Covariant Projector

$$\mathcal{P}_{\lambda\lambda'}^{\mu\nu}(p) = \varepsilon^\mu(p, \lambda)^* \varepsilon^\nu(p, \lambda')$$

master formula

$$\varepsilon^{0123} = 1$$

$$= \frac{1}{3} \left(-g^{\mu\nu} + \frac{p^\mu p^\nu}{M^2} \right) \delta_{\lambda\lambda'} - \frac{i}{2M} \epsilon^{\mu\nu\alpha\beta} p_\alpha n_\beta^i (S_i)_{\lambda\lambda'} - \frac{1}{2} n_i^\mu n_j^\nu (S_{ij})_{\lambda\lambda'}$$

$S_i, i \in \{1, 2, 3\} \rightarrow$ rotation matrices for spin-1 particle

H.S. Song, *Lett. Nuovo Cim.* 25 (1979)
S.Y. Choi, T. Lee, H.S. Song, *PRD* 40 (1989)
Fabbrichesi, Floreanini, EG, Marzola,
2302.00683 [hep-ph]

$$S_{ij} = S_i S_j + S_j S_i - \frac{4}{3} \mathbb{1} \delta_{ij} \quad (\text{see backup slides})$$

basis correspondence for $(S_i)_{\lambda\lambda'}$ $|+\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad |0\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad |-\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ corresponding to eigenevalues $\lambda = \pm 1, 0$

Density matrix of two spin-1 particles $V_1 V_2$

$$\bar{q}(p_1) q(p_2) \rightarrow V_1(k_1, \lambda_1) V_2(k_2, \lambda_2)$$

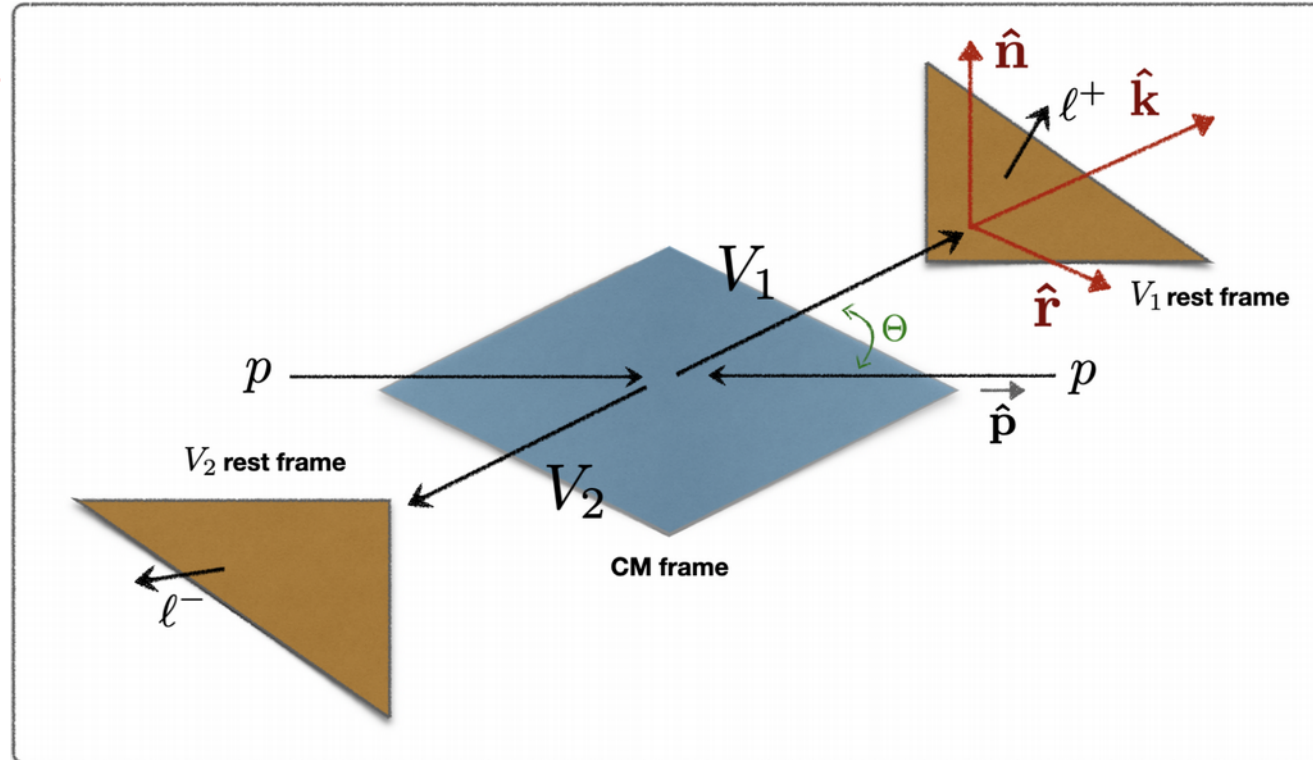
in the center of mass frame

Covariant polarization vectors

$$n_m^\mu(1) \rightarrow V_1$$

$$n_m^\mu(2) \rightarrow V_2$$

$m = 1, 2, 3$



$$\hat{\mathbf{r}} = \frac{1}{\sin \Theta} (\hat{\mathbf{p}} - \cos \Theta \hat{\mathbf{k}}), \quad \hat{\mathbf{n}} = \frac{1}{\sin \Theta} (\hat{\mathbf{p}} \times \hat{\mathbf{k}})$$

scattering angle

quarks assumed massless

(same mass)

$$n_1^\mu(1) = n_1^\mu(2) = (0, \hat{\mathbf{n}}), \quad n_2^\mu(1) = n_2^\mu(2) = (0, \hat{\mathbf{r}})$$

$$n_3^\mu(1) = \gamma(\beta, \hat{\mathbf{k}}), \quad n_3^\mu(2) = \gamma(-\beta, \hat{\mathbf{k}}),$$

β = velocity in c.m. frame

Matrix element of two-Vector Boson production

$$\bar{q}(p_1) q(p_2) \rightarrow V_1(k_1, \lambda_1) V_2(k_2, \lambda_2)$$

$$\mathcal{M}(\lambda_1, \lambda_2) = \mathcal{M}_{\mu\nu} \varepsilon^{\mu*}(k_1, \lambda_1) \varepsilon^{\nu*}(k_2, \lambda_2)$$

Density matrix

$$\rho(\lambda_1, \lambda'_1, \lambda_2, \lambda'_2) = \frac{\mathcal{M}_{\mu\nu} \mathcal{M}_{\mu'\nu'}^\dagger \mathcal{P}_{\lambda_1\lambda'_1}^{\mu\mu'}(k_1) \mathcal{P}_{\lambda_2\lambda'_2}^{\nu\nu'}(k_2)}{|\overline{\mathcal{M}}|^2}$$

9 x 9 matrix

↓ unpolarized matrix element square

$$\mathcal{P}_{\lambda\lambda'}^{\mu\nu}(p) = \varepsilon^\mu(p, \lambda)^* \varepsilon^\nu(p, \lambda')$$

↓ use the covariant expression

ρ depends on scalar products of $n_m^\mu(1)$, $n_m^\mu(2)$
with the momenta of the reaction with $m = 1, 2, 3$

Density matrix for two-QUBITS

(WW, ZZ, WZ)

decompose the **density matrix** on the basis of tensor products of Gell-Mann matrices

$$\{\mathbb{1} \otimes \mathbb{1}, \mathbb{1} \otimes T^a, T^a \otimes \mathbb{1}, T^a \otimes T^b\} \quad T^a \rightarrow 3 \times 3 \text{ Gell-Mann matrices}$$

given by the Kronecker product of the matrix representations $[A \otimes B]_{ii'jj'} = A_{ii'} B_{jj'}$

$$\rho(\lambda_1, \lambda'_1, \lambda_2, \lambda'_2) = \left(\frac{1}{9} [\mathbb{1} \otimes \mathbb{1}] + \sum_a f_a [\mathbb{1} \otimes T^a] + \sum_a g_a [T^a \otimes \mathbb{1}] + \sum_{ab} h_{ab} [T^a \otimes T^b] \right)_{\lambda_1 \lambda'_1, \lambda_2 \lambda'_2}$$

9x9 matrix

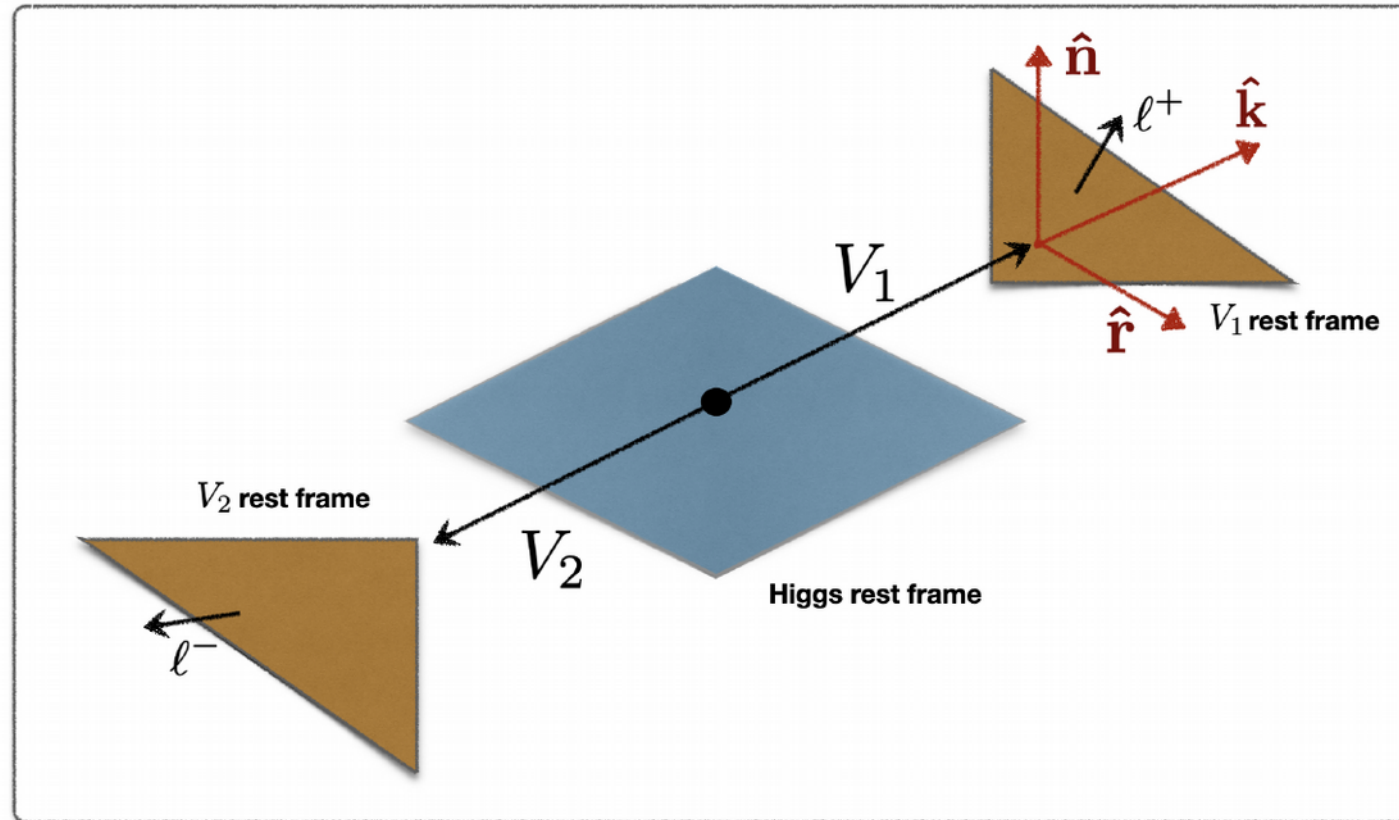
$$f_a = \frac{1}{6} \text{Tr} [\rho (\mathbb{1} \otimes T^a)] , \quad g_a = \frac{1}{6} \text{Tr} [\rho (T^a \otimes \mathbb{1})] , \quad h_{ab} = \frac{1}{4} \text{Tr} [\rho (T^a \otimes T^b)]$$

- these are scalar quantities that depend on **VV' invariant mass** and **scattering angle** Θ in c.m. frame
- we can also extract them from data, using the decay products of final VB

Phenomenological implications →

Di-boson production in Higgs boson decays

$$H \rightarrow V(k_1, \lambda_1) V^*(k_2, \lambda_2)$$

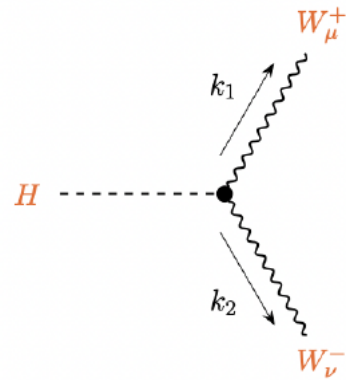
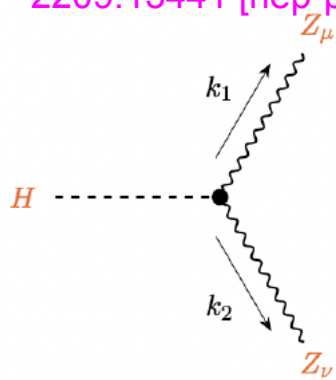


In the **Higgs boson rest frame** the density matrix of the bipartite **VV^*** system does not depend on the scattering angle, but only by the **Higgs mass**, the **V mass** and the off-shell **V^* mass**

Di-boson production in Higgs boson decays

A. Barr, PLB 825 (2022) ,
2106.01377 [hep-ph]

Aguilar-Saavedra, Bernal,
Casas, Moreno
2209.13441 [hep-ph]



$$H \rightarrow V(k_1, \lambda_1) V^*(k_2, \lambda_2)$$

Results based on
arXiv: 2302.00683 [hep-ph]

$$\left\{ \begin{array}{l} \xi_W = 1, \text{ and } \xi_Z = 1/(2c_W), \text{ with } c_W = \cos \theta_W \\ g \text{ is the weak coupling} \end{array} \right.$$

↓
Weinberg angle

V^* regarded as an off-shell vector boson with mass $M_V^* = f M_V$ $0 < f < 1$
mass of V boson M_V

Quantum Amplitude

$$\mathcal{M}_H(\lambda_1, \lambda_2) = g M_V \xi_V g_{\mu\nu} \varepsilon^{\mu*}(k_1, \lambda_1) \varepsilon^{\nu*}(k_2, \lambda_2)$$

$$\mathcal{M}_H(\lambda_1, \lambda_2) \mathcal{M}_H(\lambda'_1, \lambda'_2)^\dagger = g^2 M_V^2 \xi_V^2 g_{\mu\nu} g_{\mu'\nu'} \mathcal{P}_{\lambda_1 \lambda'_1}^{\mu\mu'}(k_1) \mathcal{P}_{\lambda_2 \lambda'_2}^{\nu\nu'}(k_2)$$

By projecting into the Gell-Mann matrix basis for 2-qutrits one can extract the **f**, **g** and **h** coefficients

Unpolarized square amplitude

$$|\overline{\mathcal{M}}_H|^2 = \frac{g^2 \xi_V^2}{4f^2 M_V^2} \left[m_H^4 - 2(1 + f^2)m_H^2 M_V^2 + (1 + 10f^2 + f^4)M_V^4 \right]$$

The non-vanishing **f,g** and **h** elements

$$g_a = f_a \text{ for } a \in \{1, \dots, 8\}$$

$$f_3 = \frac{1}{6} \frac{-m_H^4 + 2(1 + f^2)m_H^2 M_V^2 + (1 - f^2)^2 M_V^4}{m_H^4 - 2(1 + f^2)m_H^2 M_V^2 + (1 + 10f^2 + f^4)M_V^4},$$

$$f_8 = -\frac{1}{\sqrt{3}} f_3,$$

$$h_{16} = h_{61} = h_{27} = h_{72} = \frac{f M_V^2 (-m_H^2 + (1 + f^2)M_V^2)}{m_H^4 - 2(1 + f^2)m_H^2 M_V^2 + (1 + 10f^2 + f^4)M_V^4},$$

$$h_{33} = \frac{1}{4} \frac{(m_H^2 - (1 + f^2)M_V^2)^2}{m_H^4 - 2(1 + f^2)m_H^2 M_V^2 + (1 + 10f^2 + f^4)M_V^4},$$

$$h_{38} = h_{83} = -\frac{1}{4\sqrt{3}}$$

$$h_{44} = h_{55} = \frac{2f^2 M_V^4}{m_H^4 - 2(1 + f^2)m_H^2 M_V^2 + (1 + 10f^2 + f^4)M_V^4},$$

$$h_{88} = \frac{1}{12} \frac{m_H^4 - 2(1 + f^2)m_H^2 M_V^2 + (1 - 14f^2 + f^4)M_V^4}{m_H^4 - 2(1 + f^2)m_H^2 M_V^2 + (1 + 10f^2 + f^4)M_V^4},$$

Inserting the **f,g** and **h** coefficients into the Gell-Mann basis for 2-qutrits

$$\rho_H = 2 \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & h_{44} & 0 & h_{16} & 0 & h_{44} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & h_{16} & 0 & 2h_{33} & 0 & h_{16} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & h_{44} & 0 & h_{16} & 0 & h_{44} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\text{Tr} [\rho_H] = 1$$

density matrix is idempotent

$$\rho_H^2 = \rho_H$$

Signaling that $H \rightarrow VV^*$ is a **pure state**

$$\rho_H = |\Psi_H\rangle\langle\Psi_H|$$

using the basis

$$|\lambda \lambda'\rangle = |\lambda\rangle \otimes |\lambda'\rangle \text{ with } \lambda, \lambda' \in \{+, 0, -\}$$

where the pure state is

arXiv: 2302.00683 [hep-ph]

$$|\Psi_H\rangle = \frac{1}{\sqrt{2 + \varkappa^2}} [|+-\rangle - \varkappa |00\rangle + |-+\rangle]$$

$$\varkappa = 1 + \frac{m_H^2 - (1 + f)^2 M_V^2}{2f M_V^2}$$

**Aguilar-Saavedra, Bernal,
Casas, Moreno**
2209.13441 [hep-ph]

Bell's inequality for two-qutrits

- consider the following correlator $\mathcal{I}_3$ for probability measurements

Collins, Gisin, Linden,
Massar, Popescu, PRL 88 (2002)

$$\mathcal{I}_3 = \text{Tr} [\rho \mathcal{B}]$$

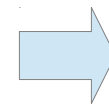
with $\mathcal{B}$ a suitable Bell operator

Optimized Bell's inequality for qutrits (CGLMP)

For deterministic local models

CGLMP

$$\mathcal{I}_3 \leq 2$$



QM qutrits can violate this inequality

with upper bound $(\mathcal{I}_3)^{\max} = 4$

- For the case of maximally entangled state $\rho = |\Psi_+\rangle\langle\Psi_+|$ optimal choice of measurements has been found $\rightarrow$ giving a specific form of $\mathcal{B}$

Acin, Durt, Gisin, Latorre,
PRA 65 (2002), quant-ph/0111143

- optimization of the Bell violation not found in general case (contrary to the case of two-qubit)
- still freedom to modify measured observables through unitary transformations U, V on $\mathcal{B}$

$$\mathcal{B} \rightarrow (U \otimes V)^\dagger \cdot \mathcal{B} \cdot (U \otimes V)$$



U, V unitary 3x3 matrices
depending on the kinematic
of the process

in order to maximize the violation of CGLMP Bell's inequality for two-qutrits

Bell's inequalities for two-qubits

correlation matrix 3x3

unit 3-dim vectors $\hat{n}_{1,2,3,4}$

which correspond to the choices that Bob and Alice do for the axis where to measure the spin

$$\left| \hat{n}_1 \cdot C \cdot (\hat{n}_2 - \hat{n}_4) + \hat{n}_3 \cdot C \cdot (\hat{n}_2 + \hat{n}_4) \right| \leq 2$$

No need of suitable choice of $\hat{n}_{1,2,3,4}$ to maximize the Bell violation

maximization process optimized for a generic correlation matrix C R. Horodecki *et al*, Phys. Lett. A200 (1995)



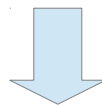
observable

$$M = C^T C$$

$$\mathfrak{m}_{12}[C] \equiv m_1 + m_2$$

the largest two eigenvalues of M

$$m_1 \geq m_2 \geq m_3$$



two-qubit state violates Bell's inequalities
IF AND ONLY IF

$$\mathfrak{m}_{12}[C] > 1$$

independently of $\hat{n}_{1,2,3,4}$

Quantifying entanglement

- **Concurrence** $\mathcal{C}[\rho]$ vanishes for separable bipartite (A,B) systems (see backup slides for definition)
Rungta, Buzek, Caves, Hillery, Milburn, PRA 64 (2001)

For qutrits

- analytical solution exists only for the lower bound

$$(\mathcal{C}[\rho])^2 \geq \mathcal{C}_2[\rho]$$

Mintert, Buchleitner,
PRL 98 (2007)



$$\mathcal{C}_2[\rho] = 2 \max \left(0, \text{Tr} [\rho^2] - \text{Tr} [(\rho_A)^2], \text{Tr} [\rho^2] - \text{Tr} [(\rho_B)^2] \right)$$

- If **non-vanishing** unequivocally signal the presence of entanglement (witness of entanglement) → used in our analysis of **entanglement** for **WW**, **ZZ** and **WZ** productions

- Solution for upper bound

$$(\mathcal{C}[\rho])^2 \leq 2 \min \left(1 - \text{Tr} [(\rho_A)^2], 1 - \text{Tr} [(\rho_B)^2] \right)$$

- **maximum value** of Concurrence can be obtained for the maximum symmetric state

for two-qutrits $\rightarrow \mathcal{C}[|\Psi_+\rangle] = 2/\sqrt{3}$ corresponding to the state $|\Psi_+\rangle = \frac{1}{\sqrt{3}} \sum_{i=1}^3 |i\rangle \otimes |i\rangle$

- On the Gell-Mann basis, **lower bound of Concurrence** can be easily computed

$$\mathcal{C}_2 = 2 \max \left[-\frac{2}{9} - 12 \sum_a f_a^2 + 6 \sum_a g_a^2 + 4 \sum_{ab} h_{ab}^2, \right. \\ \left. -\frac{2}{9} - 12 \sum_a g_a^2 + 6 \sum_a f_a^2 + 4 \sum_{ab} h_{ab}^2, 0 \right],$$

- If the bipartite (A,B) system is **a pure state** (as in the $H \rightarrow VV$ case)
it is possible to **quantify** its entanglement by computing

Entropy of entanglement

$$\mathcal{E}[\rho] = -\text{Tr} [\rho_A \log \rho_A] = -\text{Tr} [\rho_B \log \rho_B]$$

- in terms of the **von Neumann entropy** of either the two component subsystems A and B with corresponding reduced polarization density submatrices ρ_A and ρ_B

(see also backup slides)

$$0 \leq \mathcal{E}[\rho] \leq \ln 3$$

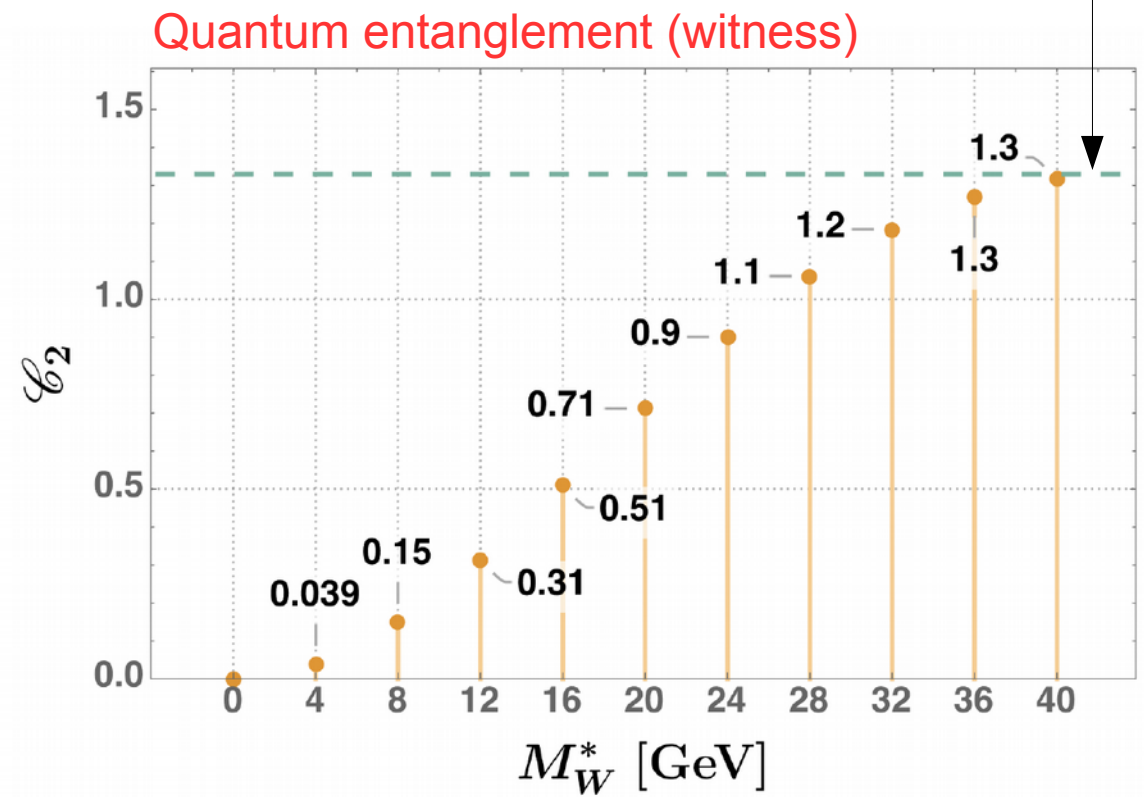
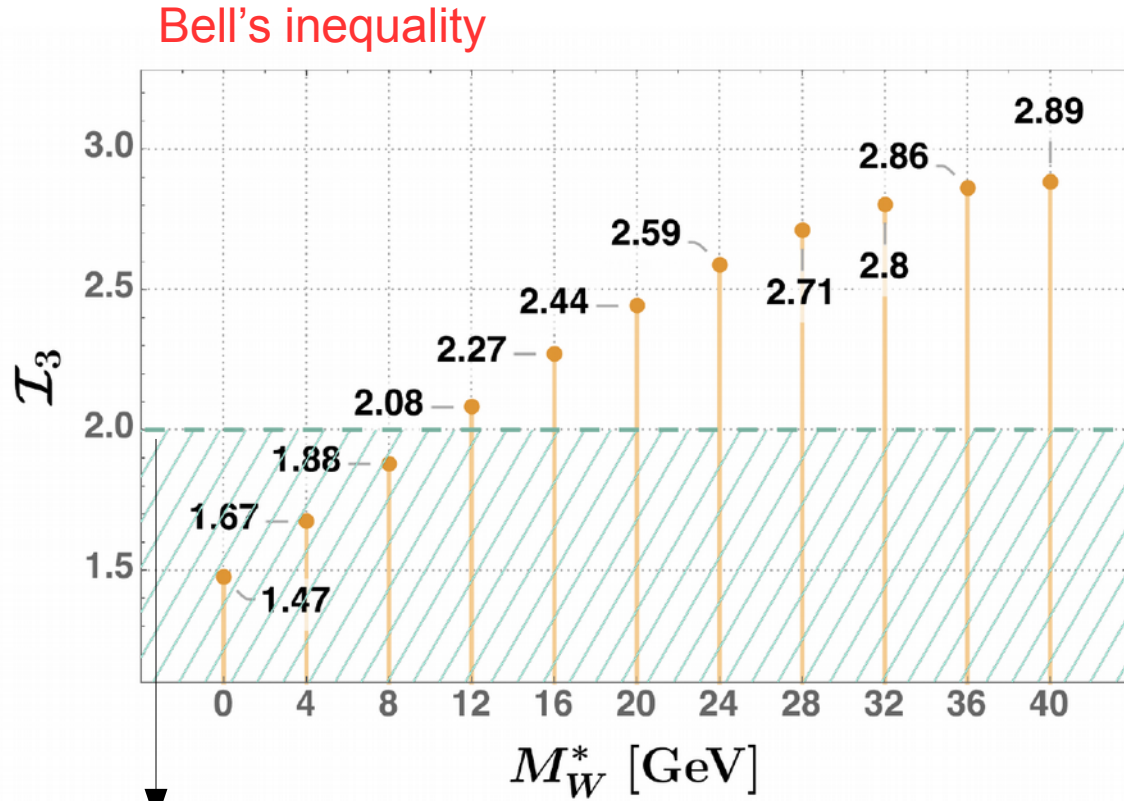
for a **two-qutrit system**

equality holds if and only
if the bipartite is separable

corresponding to the
maximally entangled state

$$H \rightarrow W W^*$$

maximum value 4/3
for a pure state



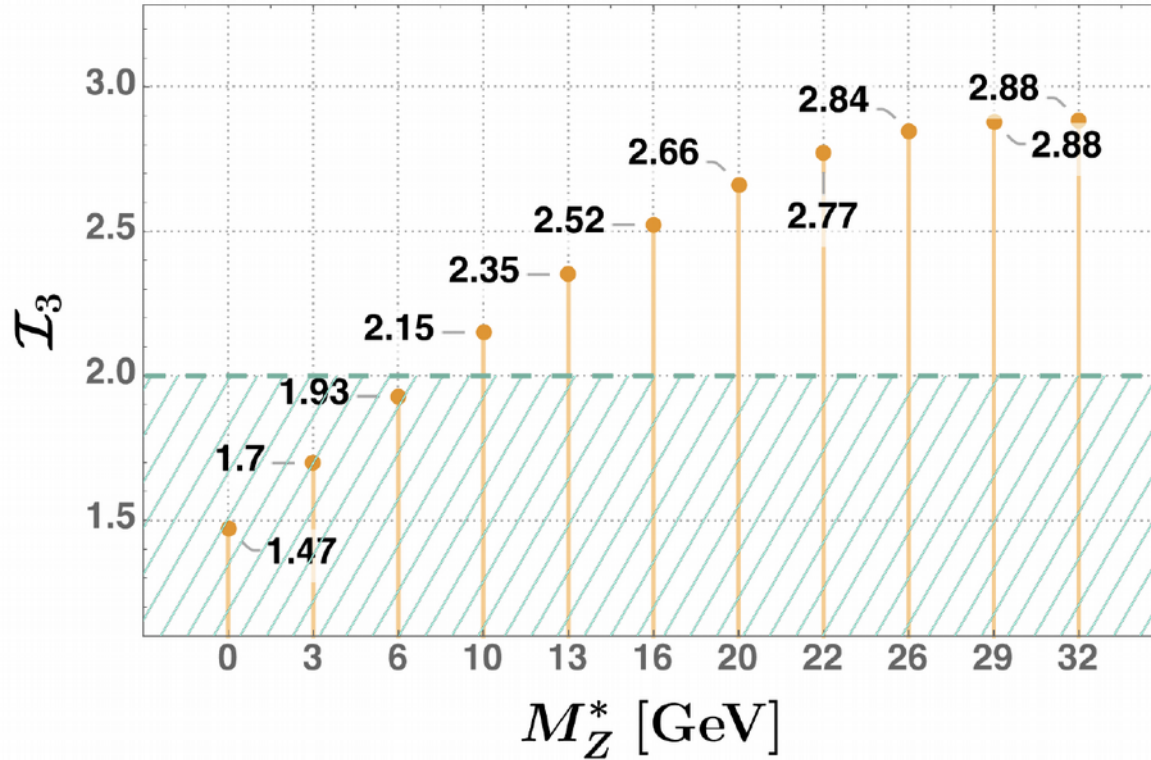
Bell's inequality violation condition $\mathcal{I}_3 > 2$

Maximization of $\mathcal{I}_3$ performed point by point as function of M_{W^*}
(see backup slides for solution of “optimized” U,V matrices in the region of max entanglement)

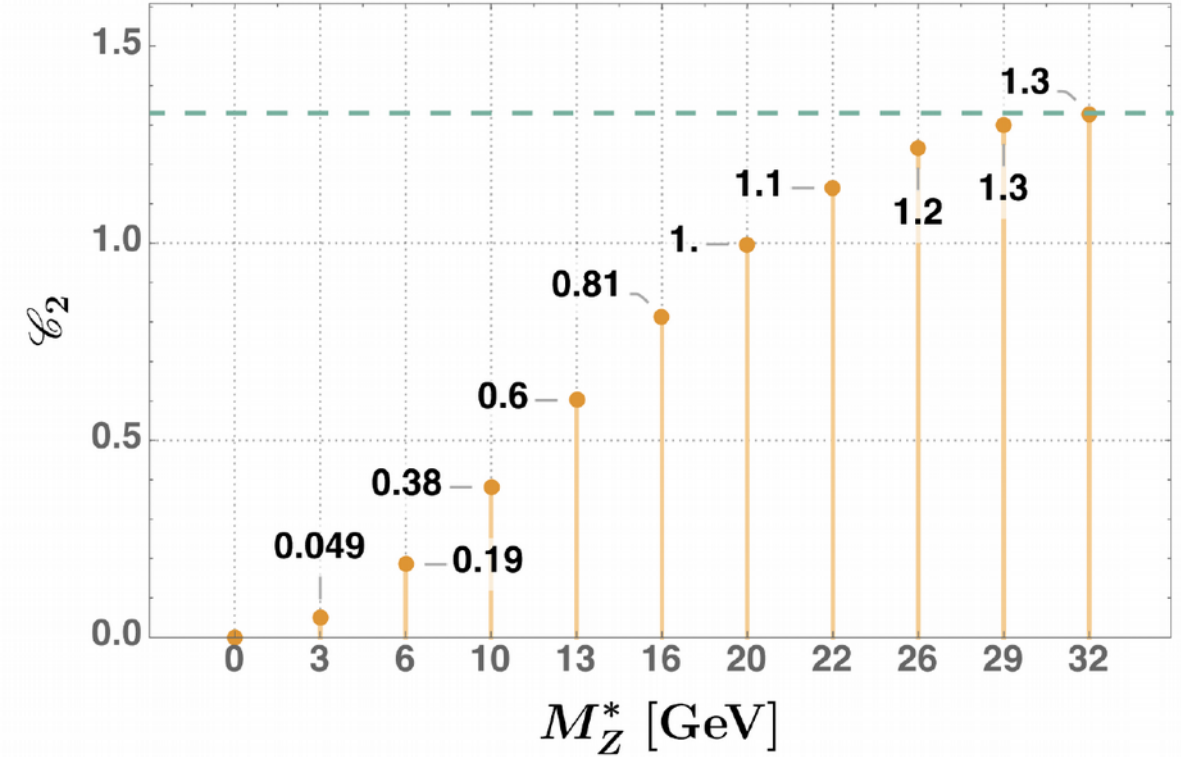
$$\mathcal{C}_2 = \frac{32f^2 M_V^4 \left[m_H^4 - 2(1 + f^2)m_H^2 M_V^2 + (1 + 4f^2 + f^4)M_V^4 \right]}{\left[m_H^4 - 2(1 + f^2)m_H^2 M_V^2 + (1 + 10f^2 + f^4)M_V^4 \right]^2}$$

$$H \rightarrow Z Z^*$$

Bell's inequality



Quantum entanglement (witness)



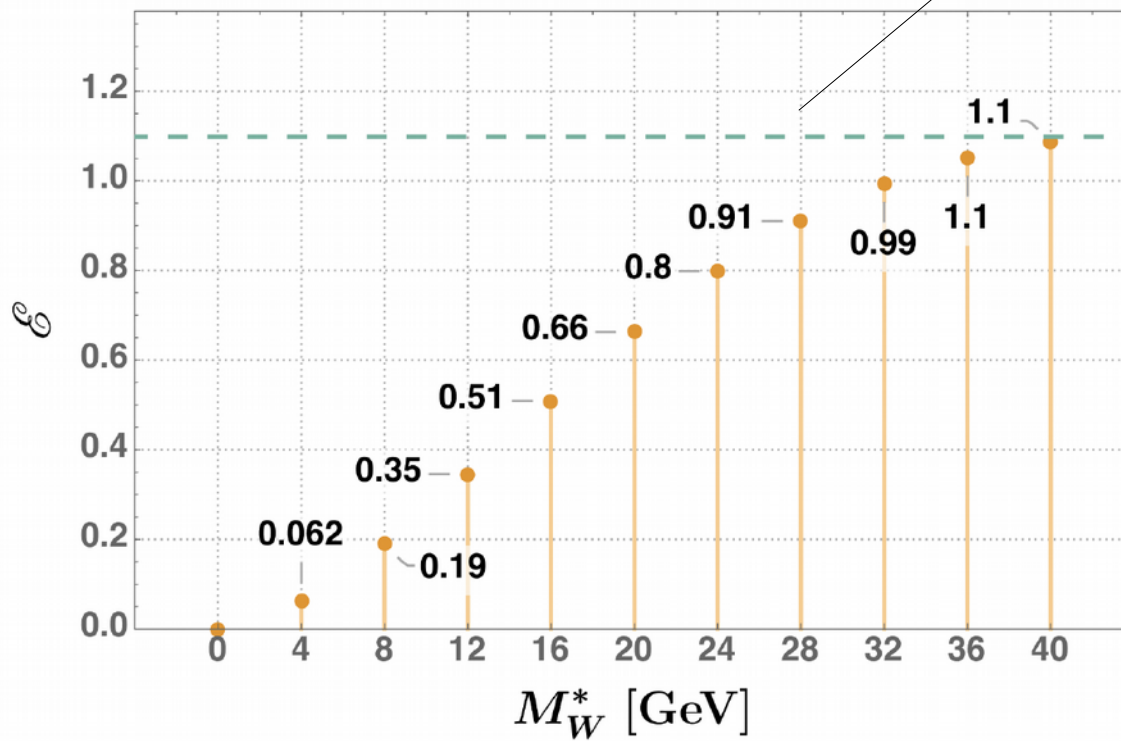
Maximization of $\mathcal{I}_3$ performed point by point as a function of M_{Z^*}
 (see backup slides for optimized U,V matrices in the region of max entanglement)

Entropy of entanglement for $H \rightarrow VV^*$

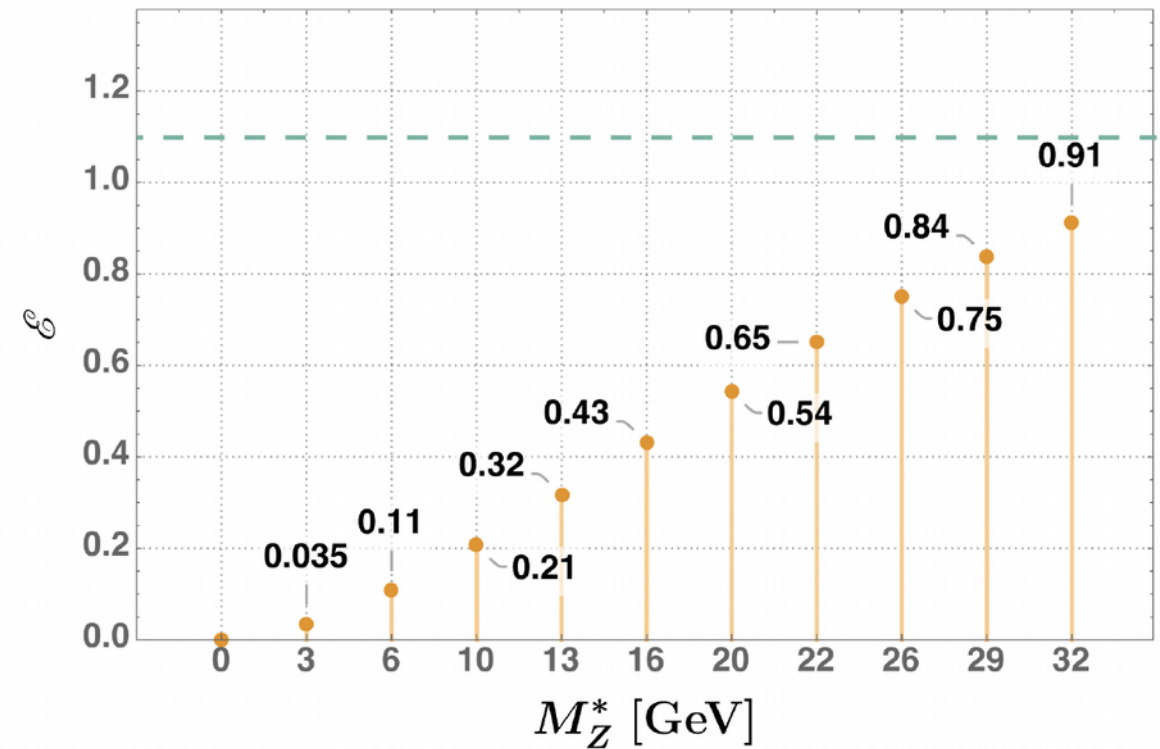
arXiv: 2302.00683 [hep-ph]

$$H \rightarrow WW^*$$

max = $\log[3]$



$$H \rightarrow ZZ^*$$



Sensitivity to the Bell violation

$$H \rightarrow WW^* \quad ZZ^*$$

		$\ell^+ \nu_\ell \ell^- \bar{\nu}_\ell$	$\ell^- \ell^+ \ell^- \ell^+$
<u>LHC run2</u>	$(\mathcal{L} = 140 \text{ fb}^{-1})$	4571	28
<u>Hi-Lumi</u>	$(\mathcal{L} = 3 \text{ ab}^{-1})$	9.8×10^3	589

cuts in V^* invariant mass: 40 GeV for WW^* and 30 GeV for ZZ^*

- we used **MADGRAPH5 @ LO** corrected by k factors at **N3LO + N3LL**
Bonvini, Marzani, Muselli, Rottoli, JHEP 08, 105 (2016) 1603.08000 [hep-ph]
- take into account **efficiency** (70%) in the identification of final leptons
- events reduced of **25% for ZZ^*** and **50% for WW^***

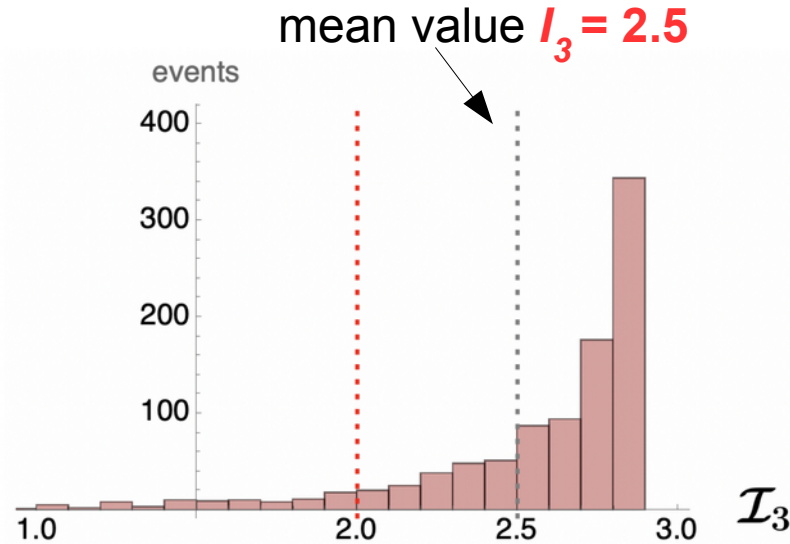
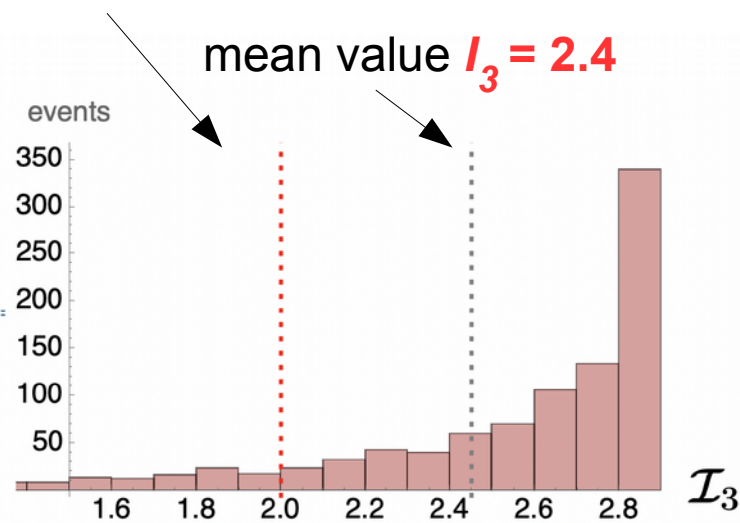
- backgrounds can be efficiently reduced after considering topology of final states and use of suitable cuts
- we neglect all background effects in our assessment of significance
- We model uncertainty as a Gaussian dispersion in the determination of the **mass of the off-shell weak boson** controlled by the number of events
- For the WW^* we add a **systematic error** taking into account uncertainty in the reconstruction of **W^* mass**, taking as benchmark value = **5 GeV** for the **systematic error** in reconstruction of neutrino momenta

$H \rightarrow WW^*$

$H \rightarrow ZZ^*$

WW* dominated by the large systematic error in reconstructing electron momenta due to neutrinos

Bell violation

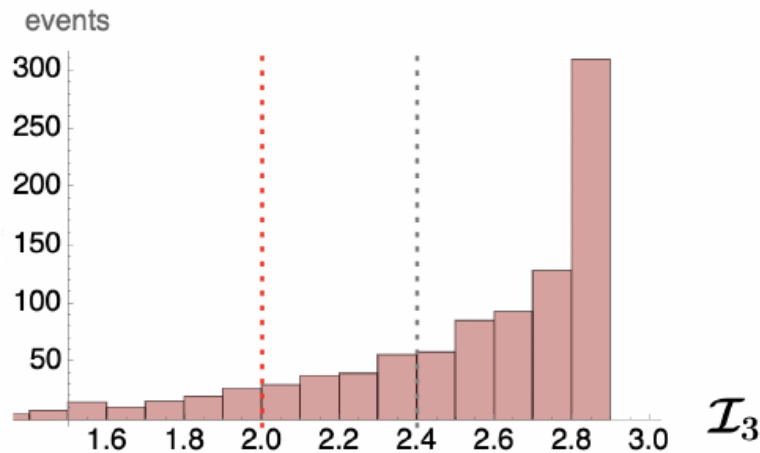


1000 pseudo experiments

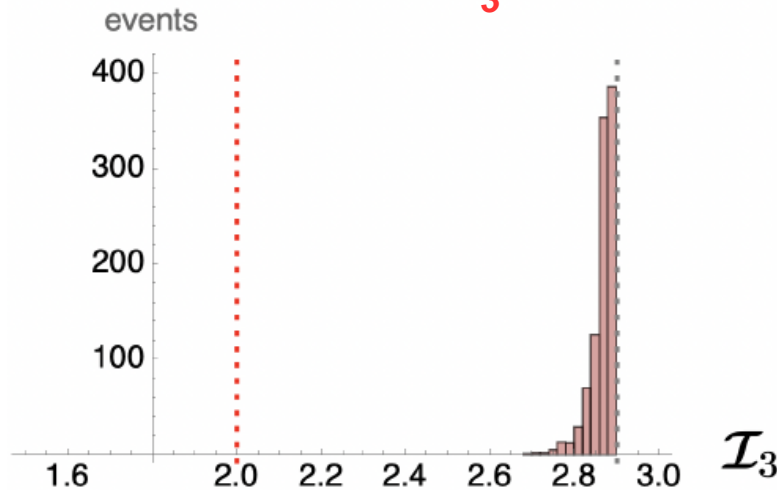
LHC (run2)

$$\mathcal{L} = 140 \text{ fb}^{-1}$$

mean value $I_3 = 2.4$



mean value $I_3 = 2.9$



LHC (Hi-lumi)

$$\mathcal{L} = 3 \text{ ab}^{-1}$$

$$\Phi(x) = \frac{1}{2} \left[1 + \text{erf} \left(\frac{x}{\sqrt{2}} \right) \right]$$

for significance $\rightarrow \mathcal{Z} = \Phi^{-1}(1 - p) \rightarrow$

N. of standard deviation in the approx. in which the distribution is assumed to be Gaussian

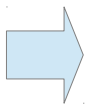
Summary of results for $H \rightarrow WW^*$ and $H \rightarrow ZZ^*$

significance S for rejecting the null hypothesis $\mathcal{I}_3 \leq 2$

	$S(WW^*) \rightarrow$ uncertainty dominated by the systematic error	$S(ZZ^*) \rightarrow$ uncertainty dominated by the statistical error
LHC Run 2	1	1.3
LHC Hi-Lumi	1	5.6

$\rightarrow$ larger due to smaller uncertainty and higher mean value (2.9)

ZZ^* promising channel at LHC Hi-Lumi

- In real simulation a **reduced significance** is expected (systematic uncertainties due to unfolding, background, and detector were neglected or only partially modeled here)
- Results are also consistent with analysis of  **A. Barr, PLB 825 (2022), 2106.01377 [hep-ph]** **Aguilar-Saavedra, Bernal, Casas, Moreno 2209.13441 [hep-ph]** where simulations of events have been adopted for the reconstruction of the density matrix

Di-boson production in pp collisions

Drell-Yan processes

$$p p \rightarrow V_1 V_2$$

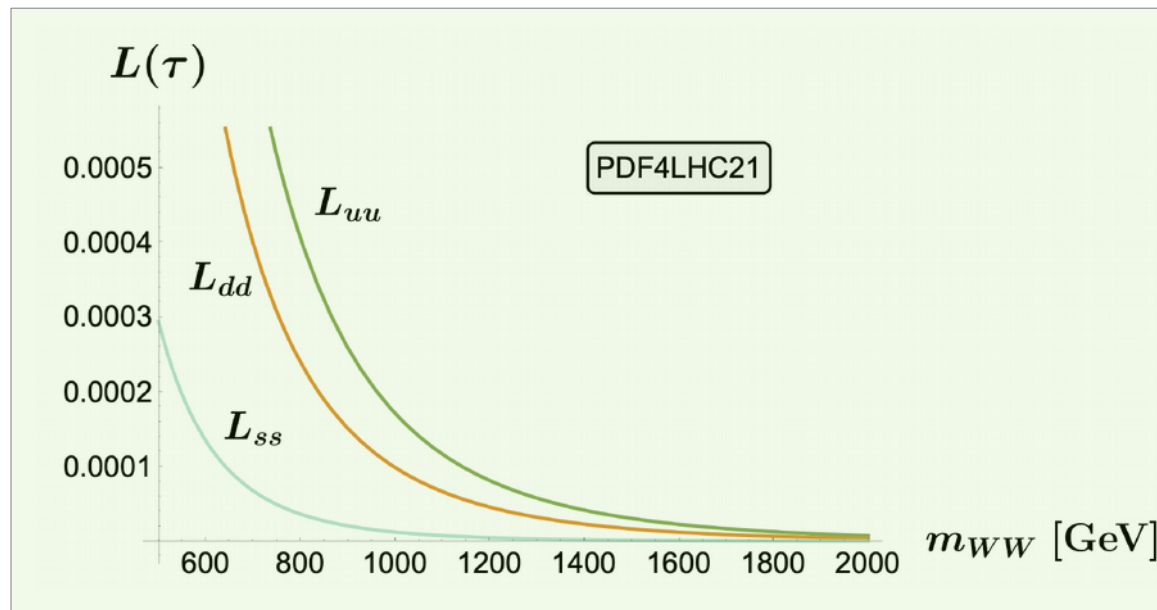
$$V_1, V_2 \in \{W, Z\}$$

Proceeds via the partonic quark-antiquark process

(away from the Higgs resonance)

$$q_1 \bar{q}_2 \rightarrow V_1 V_2$$

weighted by the parton luminosity $L^{q_1 \bar{q}_1}(\tau)$



parton luminosity of the initial $q_1 \bar{q}_2$ state

$$L^{q_1 \bar{q}_1}(\tau) = \frac{4\tau}{\sqrt{s}} \int_{\tau}^{1/\tau} \frac{dz}{z} q_{q_1}(\tau z) q_{\bar{q}_2}\left(\frac{\tau}{z}\right)$$

$\tau = m_{VV} / \sqrt{s}$ → pp c.o.m. energy
 invariant mass of $V_1 V_2$ system

Decomposing the matrix density into the Gell-Mann matrix basis

$$\rho(\lambda_1, \lambda'_1, \lambda_2, \lambda'_2) = \left(\frac{1}{9} [\mathbb{1} \otimes \mathbb{1}] + \sum_a f_a [\mathbb{1} \otimes T^a] + \sum_a g_a [T^a \otimes \mathbb{1}] + \sum_{ab} h_{ab} [T^a \otimes T^b] \right)_{\lambda_1 \lambda'_1, \lambda_2 \lambda'_2}$$

we obtain for the **h** correlations coefficients in VV production $\longrightarrow$ depend also on **scattering angle** Θ

$$h_{ab}[m_{VV}, \Theta] = \frac{\sum_{q=u,d,s} L^{q\bar{q}}(\tau) \left(\tilde{h}_{ab}^{q\bar{q}}[m_{VV}, \Theta] + \tilde{h}_{ab}^{q\bar{q}}[m_{VV}, \Theta + \pi] \right)}{\sum_{q=u,d,s} L^{q\bar{q}}(\tau) \left(A^{q\bar{q}}[m_{VV}, \Theta] + A^{q\bar{q}}[m_{VV}, \Theta + \pi] \right)}$$

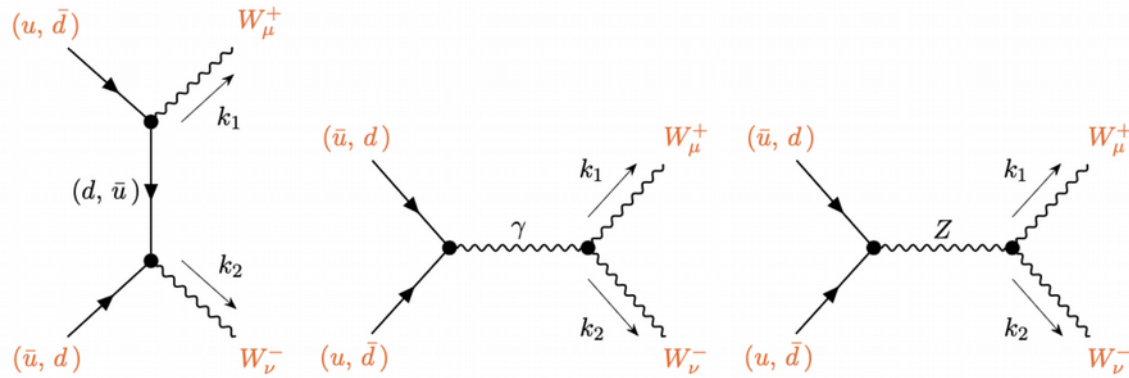
and analogously for the f_a and g_a correlation coefficients, where

$$\tilde{h}_{ab} = A^{q\bar{q}} h_{ab}$$

$A^{q\bar{q}}$ = unpolarized square amplitude of the partonic process

- main **uncertainty** on the correlation coefficients comes from the missing higher order **QCD corrections**
- giving approx a **10% uncertainty** on the main entanglement observables
- other theoretical uncertainties**, mainly from PDF and other SM inputs, is **negligible** $\rightarrow$ of the order of **permille** effect

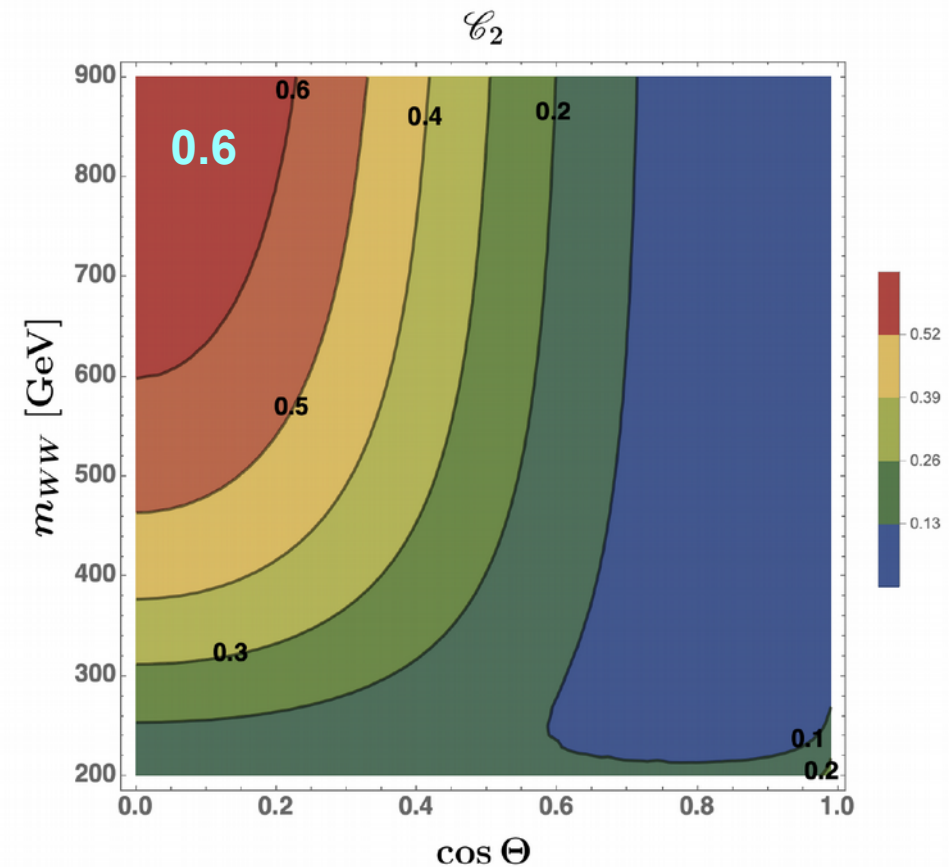
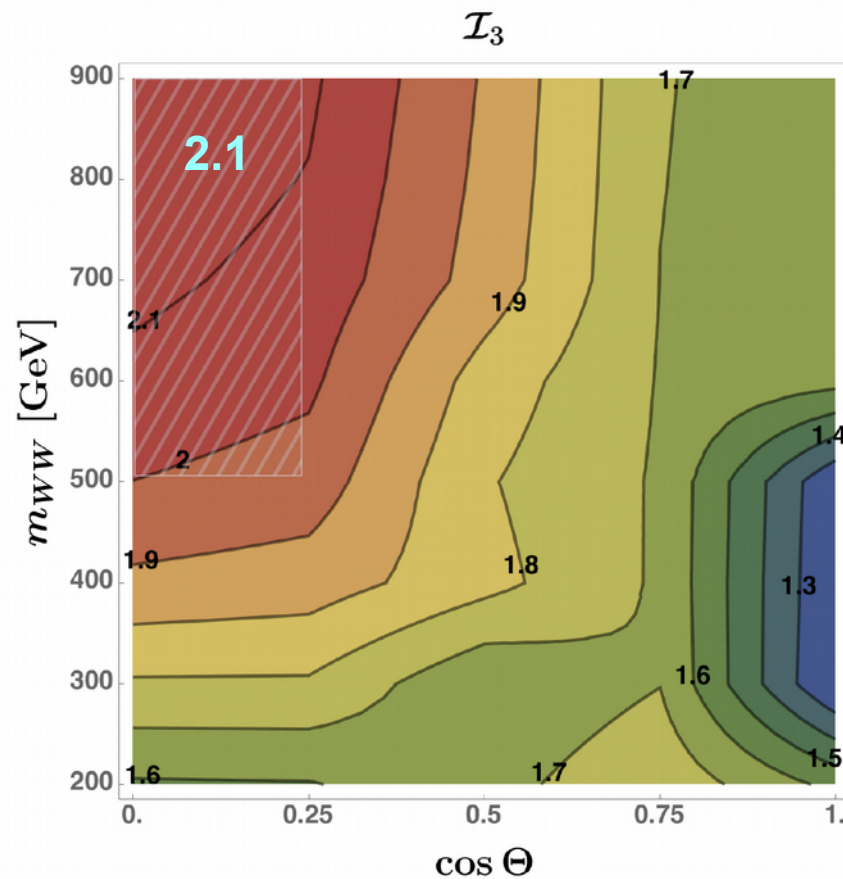
$$pp \rightarrow W^+W^-$$



optimization for **maximum value** of $\mathcal{I}_3$
of Bell's inequality violation is employed point
by point in the Θ m_{WW} space

$$\mathcal{B} \rightarrow (U \otimes V)^\dagger \cdot \mathcal{B} \cdot (U \otimes V)$$

(see backup slides for their expressions in hatched area)



hatched area in the left-plot for $\mathcal{I}_3 > 2$ indicates bin used as reference for our estimation of the significance (see next slides)

Events and sensitivity

$$p p \rightarrow W^+ W^-$$

(run2) $\mathcal{L} = 140 \text{ fb}^{-1}$

(Hi-Lumi) $\mathcal{L} = 3 \text{ ab}^{-1}$

events

288

6145

- estimated by using MADGRAPH5 @ LO for cross sections, corrected by the k-factors at the NNLO

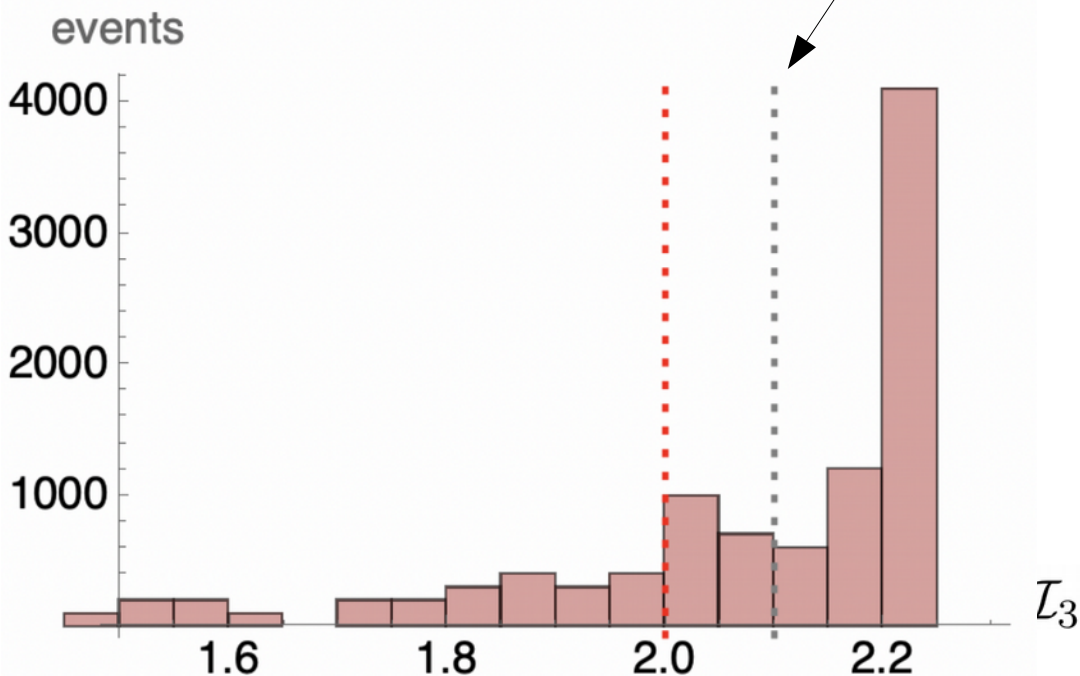
- 70% efficiency in identification of each lepton
→ number of events reduced accordingly

$m_{WW} > 500 \text{ GeV}$ and $\cos \Theta < 0.25$

- model uncertainty as a Gaussian dispersion in determination of WW invariant mass and scattering angle controlled by the N. of events
- add a systematic error with benchmark value of 5 GeV that takes into account significant uncertainty in reconstruction of neutrino momenta

Bell's inequality violation

mean value = 2.1



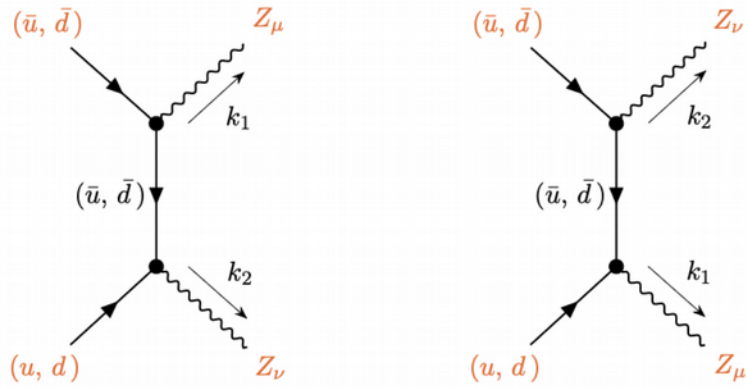
10^4 pseudo experiments

- similar plot for Hi-Lumi (statistical error dominate)

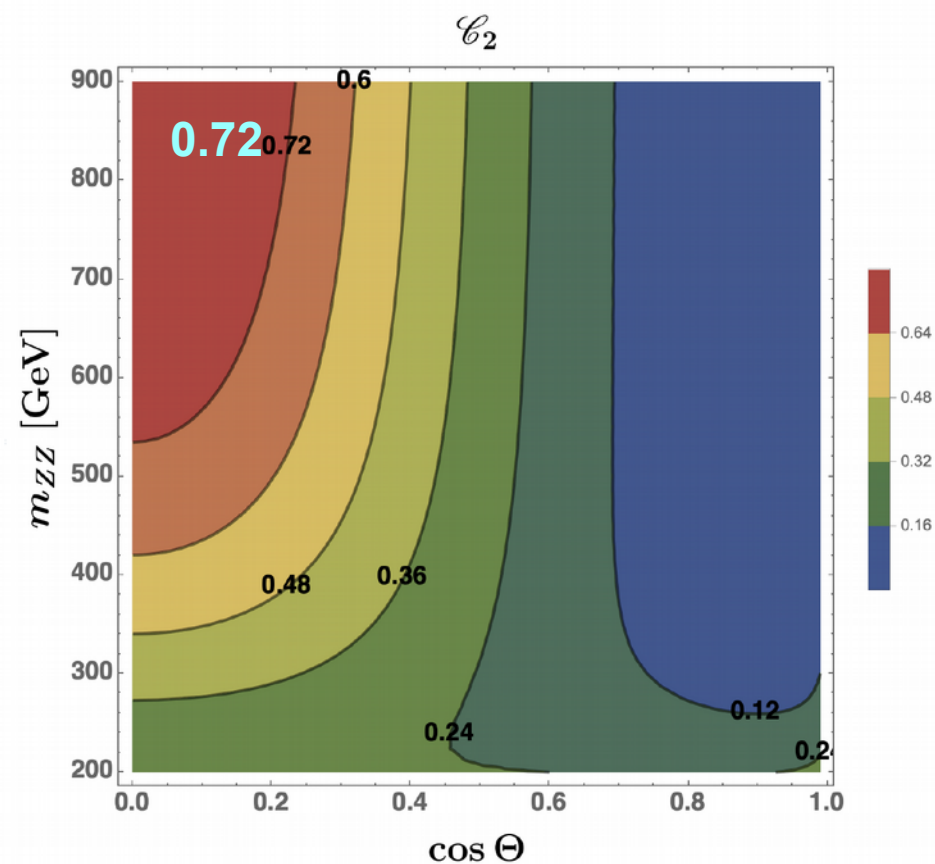
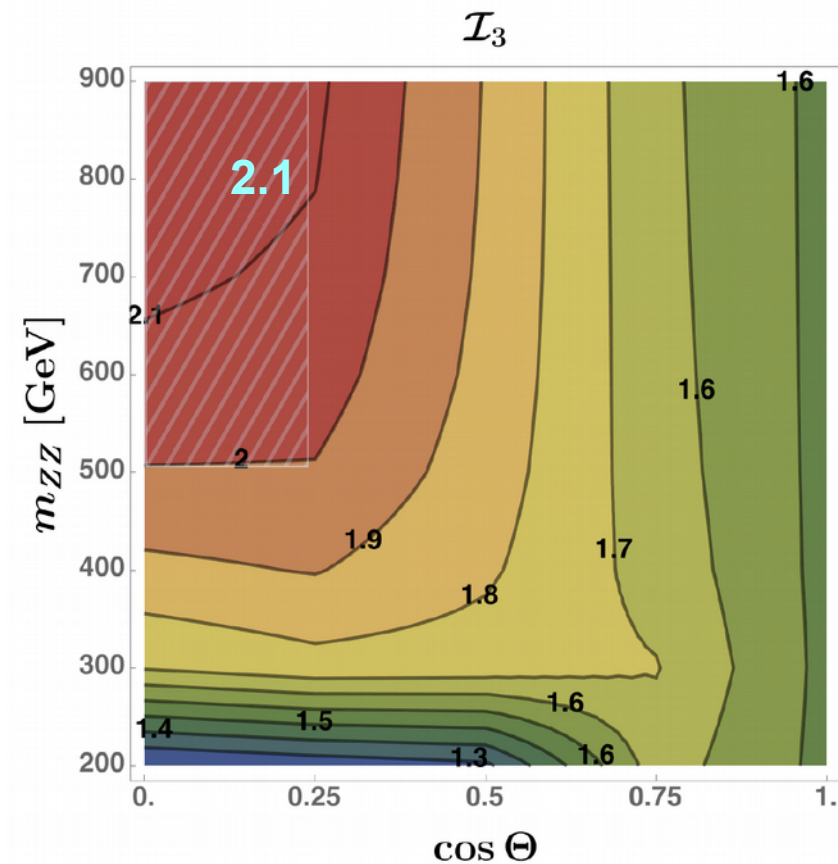
- Significance 0.8 at run 2 and similar at Hi-lumi to reject the null hypothesis $\mathcal{I}_3 \leq 2$

$$pp \rightarrow ZZ$$

arXiv: 2302.00683 [hep-ph]



- same strategy as in WW for optimization of **maximum value** of Bell's inequality violation
- reduced effect for Bell violations with respect to WW, but error is smaller being dominated by statistics



hatched area in the left-plot for $\mathcal{I}_3 > 2$ indicates bins used as reference for our estimation of the significance (see next slides)

Events and sensitivity

$$pp \rightarrow ZZ$$

(run2) $\mathcal{L} = 140 \text{ fb}^{-1}$ (Hi-Lumi) $\mathcal{L} = 3 \text{ ab}^{-1}$

events	4	77

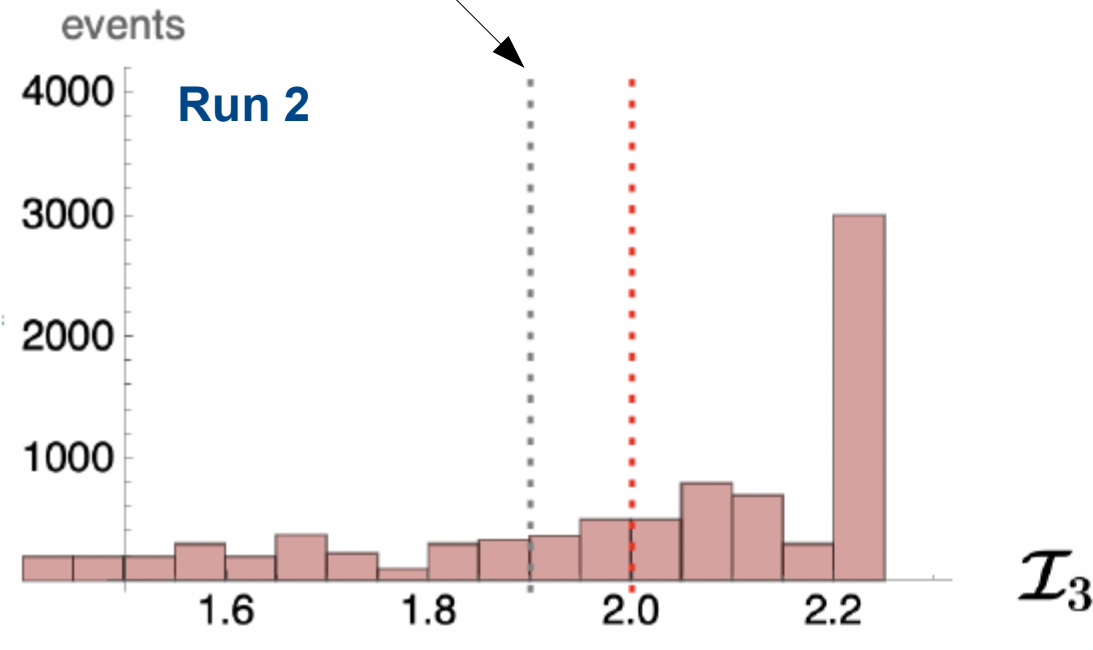
Less events than WW but statistical error dominates

$m_{ZZ} > 500 \text{ GeV}$ and $\cos \Theta < 0.25$

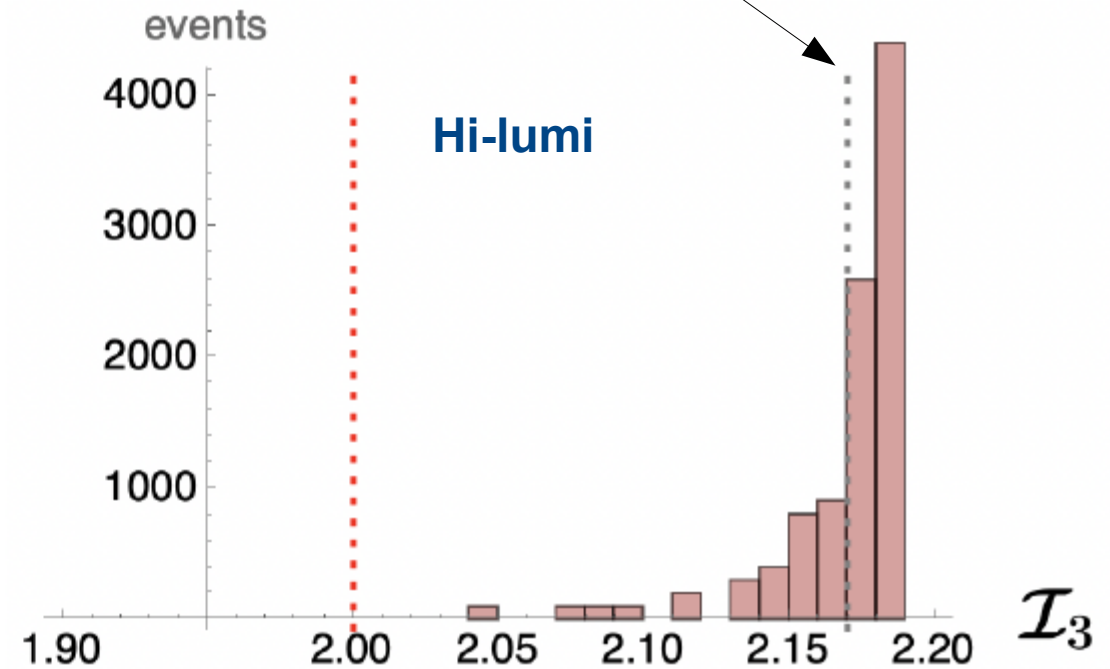
10^4 pseudo experiments

Bell's inequality violation

mean value = 1.9



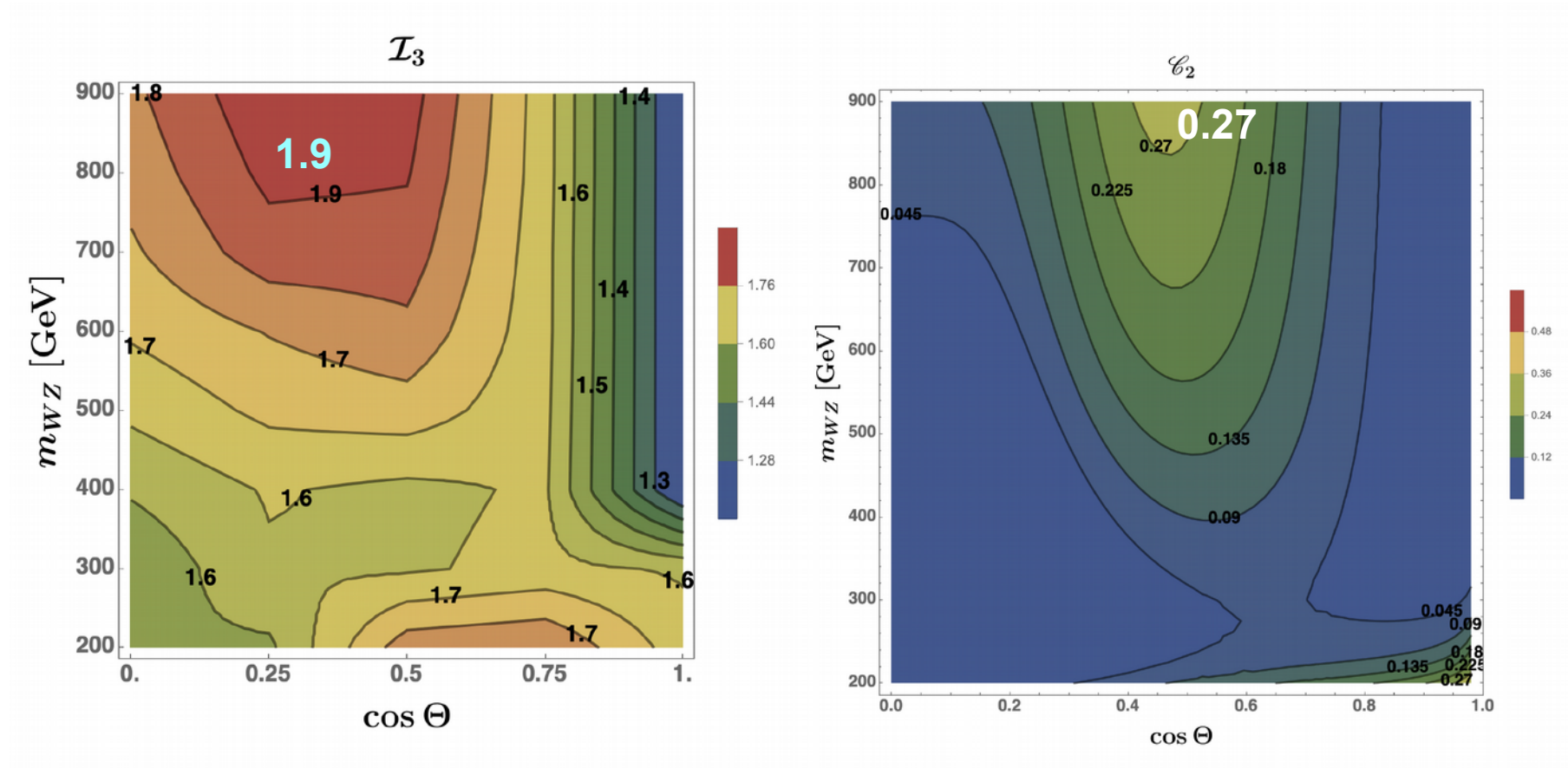
mean value = 2.2



At Hi-lumi significance more than 2 to reject the null hypothesis $\mathcal{I}_3 \leq 2$

$$p p \rightarrow W^+ Z$$

arXiv: 2302.00683 [hep-ph]

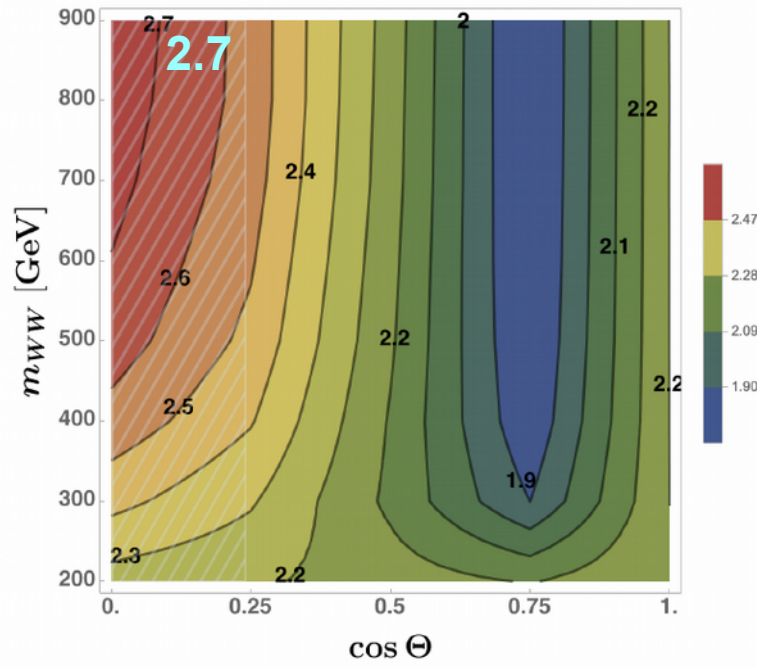
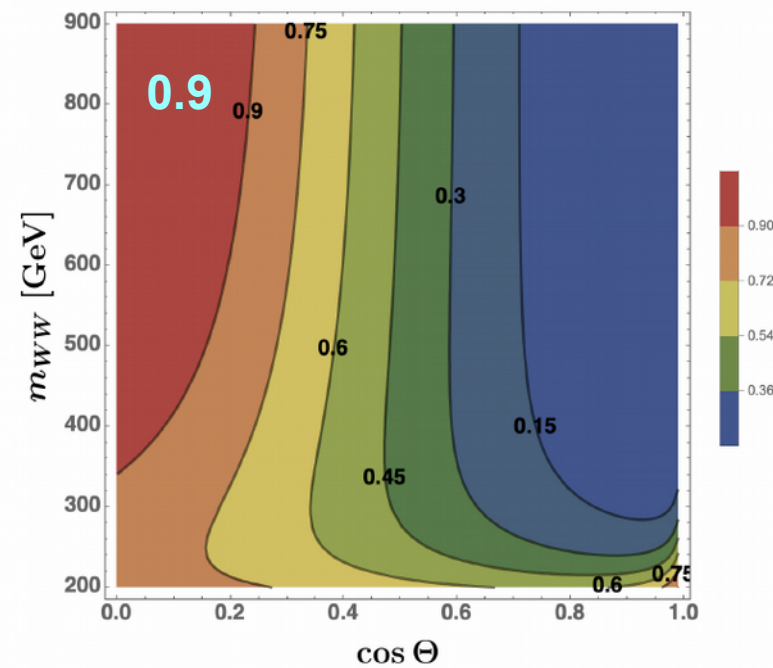


No violation of Bell's inequalities in the relevant kinematic regions ($m_{WZ} \sim 1$ TeV) .

WW, ZZ, WZ production analyzed also in [2209.13990 \[quant-ph\]](#) using events simulation at partonic level. conclusions differ from our results (possible underestimated errors..)

Lepton colliders

$$e^+ e^- \rightarrow W^+ W^-$$

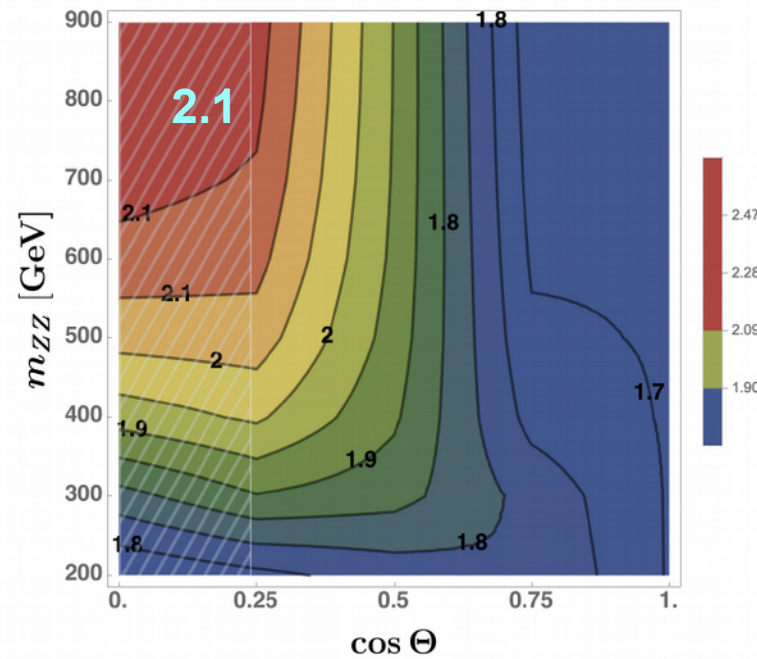
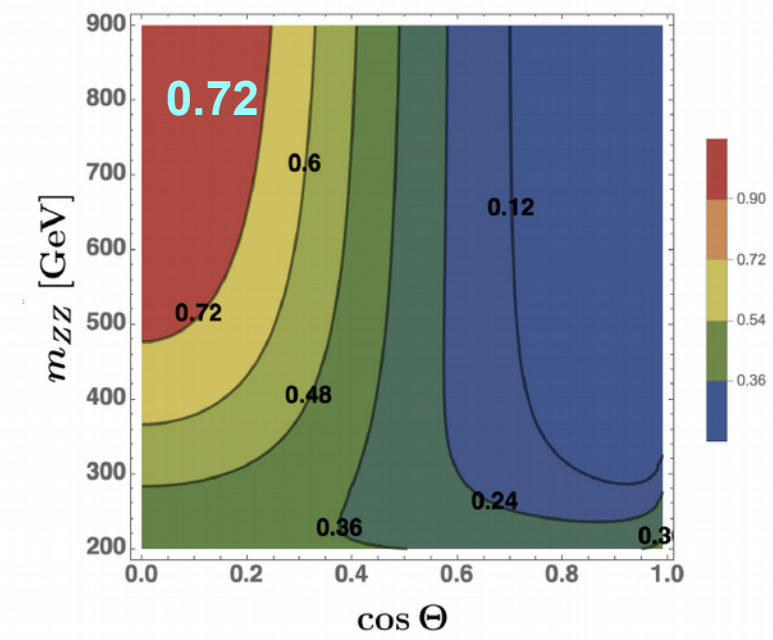
 $\mathcal{I}_3$

 $\mathcal{C}_2$


$$h_{ab}[m_{WW}, \Theta] = \frac{\tilde{h}_{ab}^{\ell\bar{\ell}}[m_{WW}, \Theta]}{A^{\ell\bar{\ell}}[m_{WW}, \Theta]},$$

$$f_a[m_{WW}, \Theta] = \frac{\tilde{f}_a^{\ell\bar{\ell}}[m_{WW}, \Theta]}{A^{\ell\bar{\ell}}[m_{WW}, \Theta]},$$

$$g_a[m_{WW}, \Theta] = \frac{\tilde{g}_a^{\ell\bar{\ell}}[m_{WW}, \Theta]}{A^{\ell\bar{\ell}}[m_{WW}, \Theta]},$$

absence of PDF increases entanglement and Bell violations in WW

 $\mathcal{I}_3$

 $\mathcal{C}_2$


$$e^+ e^- \rightarrow Z Z$$

ZZ channel not much different than LHC → due to the fact that PDF almost factorize (t,u channels)

Lepton colliders : events and sensitivity

$$e^+ e^- \rightarrow W^+ W^-$$

$$\sqrt{s} = 1 \text{ TeV}$$

$$\sqrt{s} = 368 \text{ GeV}$$

$$(\text{muon}) \mathcal{L} = 1 \text{ ab}^{-1}$$

$$(\text{FCC}) \mathcal{L} = 1.5 \text{ ab}^{-1}$$

events

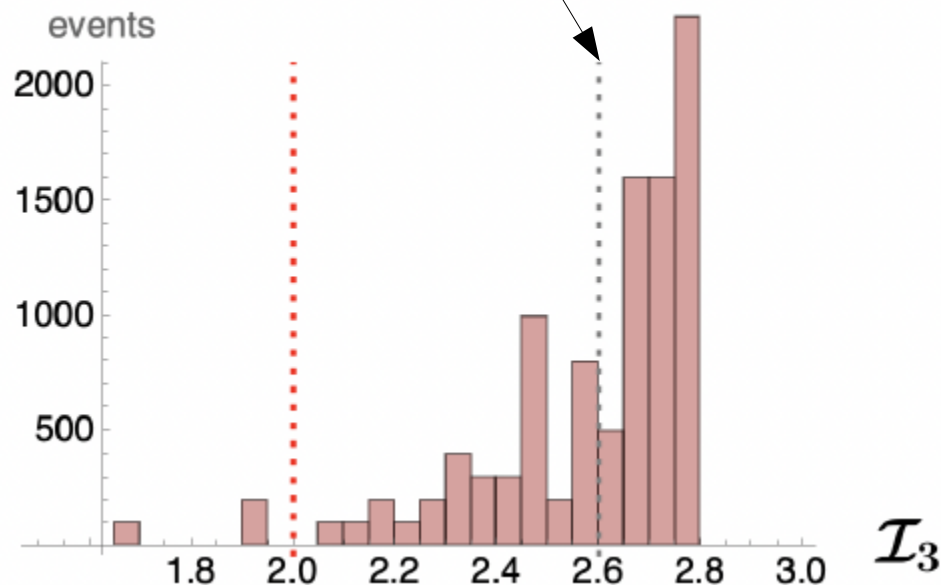
$$3.6 \times 10^3$$

$$5.8 \times 10^4$$

$$m_{WW} > 200 \text{ GeV and } \cos \Theta < 0.25$$

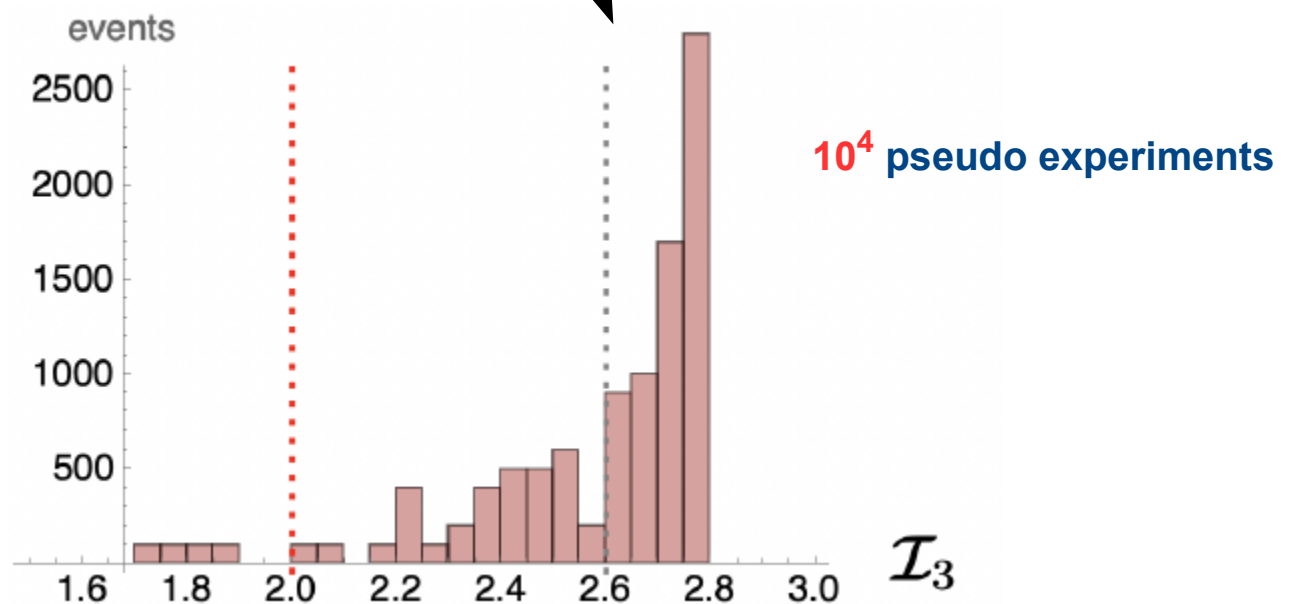
Significance = 2 to reject the null hypothesis
both at muon collider and FCC

mean value = 2.6



- Events computed with MADGRAPH@LO
- Efficiency 70% in each lepton identification
→ n. of events reduced accordingly
- statistic error due to neutrino momenta reconstruction (assumed as in $pp \rightarrow WW$)
largely dominates against statistical one
- no background considered

mean value = 2.6



Lepton colliders : events and sensitivity

$$e^+ e^- \rightarrow Z Z$$

promising channel

$$\sqrt{s} = 1 \text{ TeV}$$

$$\sqrt{s} = 368 \text{ GeV}$$

$$(\text{muon}) \mathcal{L} = 1 \text{ ab}^{-1}$$

$$(\text{FCC}) \mathcal{L} = 1.5 \text{ ab}^{-1}$$

events

44

748

$$m_{WW} > 200 \text{ GeV and } \cos \Theta < 0.25$$

Significance to reject null hypothesis

2 @ muon collider and > 4 @ FCC

Events computed with MADGRAPH@LO

Efficiency 70% in each lepton identification
→ n. of events reduced accordingly

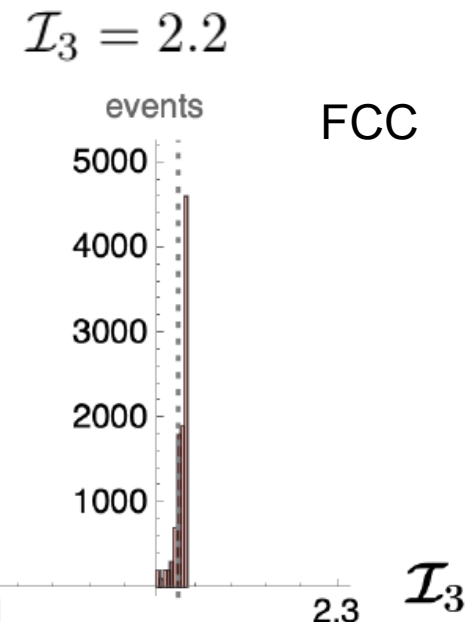
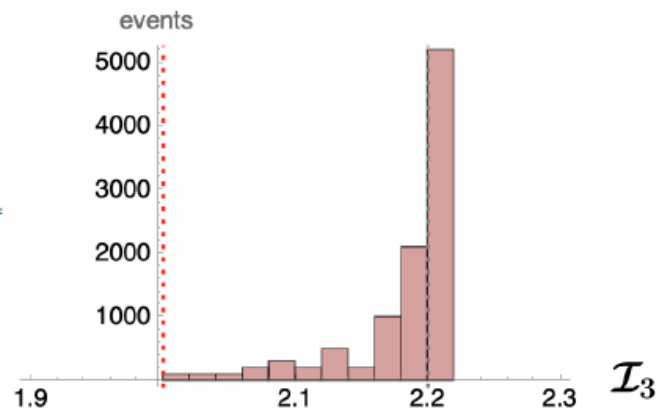
only statistic error assumed

no background considered

FCC significance is larger than muon collider due to larger n. of events at FCC (statistical error reduced)

10^4 pseudo experiments

$\mathcal{I}_3 = 2.2$
Muon collider



Constraining HWW and HZZ anomalous couplings with Quantum Tomography at the LHC

Fabbrichesi, Floreanini, EG, Marzola (preliminary)

We use polarization density matrix of the processes

$$H \rightarrow WW^*$$

$$H \rightarrow ZZ^*$$



CP-even



CP-odd

to constrain anomalous Higgs couplings to WW and ZZ

Effective Higgs- VV Lagrangian (including SM)

$$V^{\mu\nu} \rightarrow \text{Field strength, } V=W,Z$$

$$\tilde{V}^{\mu\nu} = \epsilon^{\mu\nu\alpha\beta} V_{\alpha\beta} \quad (\text{dual})$$

$$\begin{aligned} \mathcal{L}_{HVV} = & g m_W W_\mu^+ W^{-\mu} H + \frac{g}{2 \cos \theta_W} m_Z Z_\mu Z^\mu H \\ & - \frac{g}{m_W} \left[\frac{\lambda_1^W}{2} W_{\mu\nu}^+ W^{-\mu\nu} + \lambda_2^W \left(W^{+\nu} \partial^\mu W_{\mu\nu}^- + \text{H.c.} \right) + \frac{\tilde{\lambda}_{CP}^W}{4} W_{\mu\nu}^+ \tilde{W}^{-\mu\nu} \right. \\ & \left. + \frac{\lambda_1^Z}{2} Z_{\mu\nu} Z^{\mu\nu} + \lambda_2^Z Z^\nu \partial^\mu Z_{\mu\nu} + \frac{\tilde{\lambda}_{CP}^Z}{4} Z_{\mu\nu} \tilde{Z}^{\mu\nu} \right] H, \end{aligned}$$

(preliminary)

Bounds at LHC

Limits obtained by using suitable observables based on **density matrix**

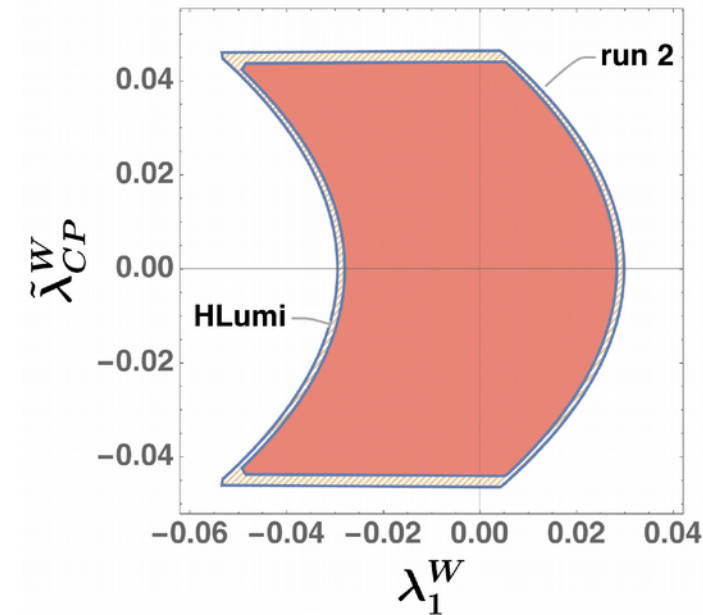
All limits are at the 95% C.L .

LHC run 2 ($\mathcal{L} = 140 \text{ fb}^{-1}$)

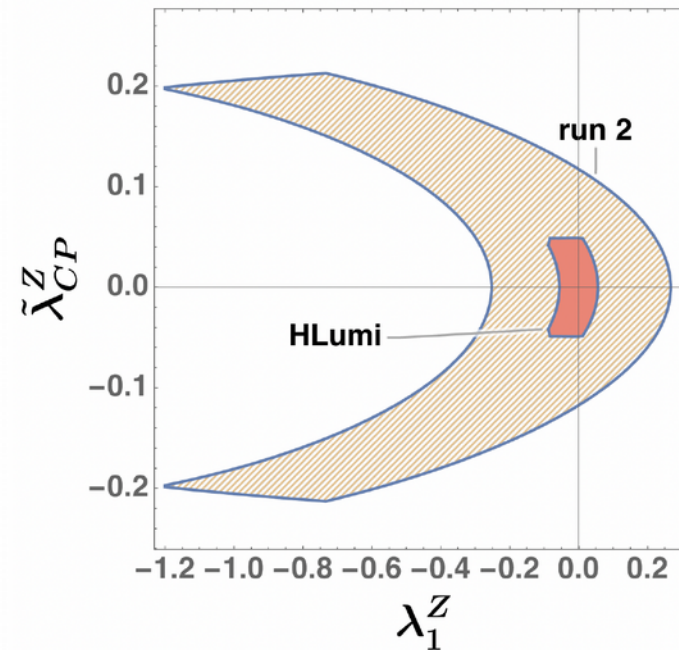
HLumi ($\mathcal{L} = 3 \text{ ab}^{-1}$)

[CMS Coll], 1901.00174 [hep-ex]

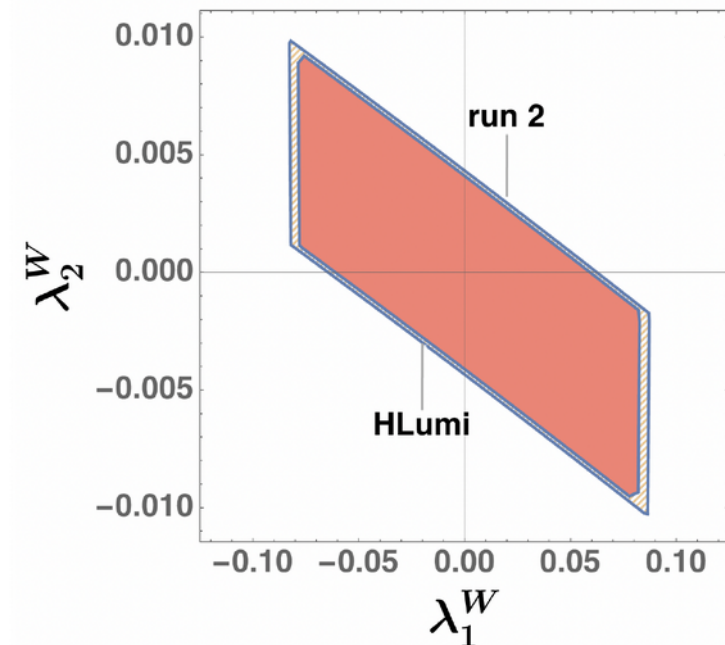
$H \rightarrow WW^*$



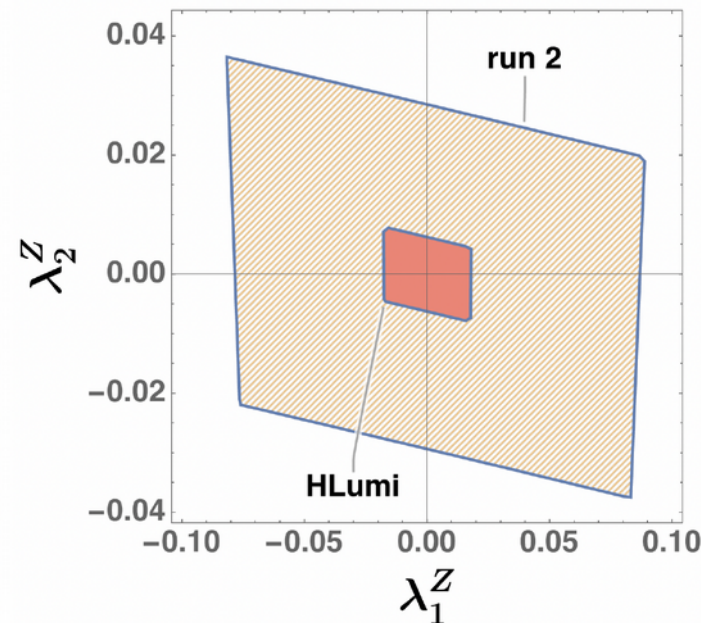
$H \rightarrow ZZ^*$



$H \rightarrow WW^*$



$H \rightarrow ZZ^*$



Limits comparable or even stronger than current experimental bounds @ LHC based on cross sections

Competitive even with projected bounds from future linear colliders **Sharma, Shivaji**, 2207.03862 [hep-ph]

Conclusions

- We determined the density matrix for $H \rightarrow WW^*, ZZ^*$ and $pp \rightarrow WW, ZZ, WZ$ processes
- analyzed the optimized CGLMP inequality and an observable (lower-bound) of entanglement provided by the $\mathcal{I}_3$ and $\mathcal{C}_2$ observables respectively
- We found that CGLMP inequality is violated in di-boson Higgs decay and $H \rightarrow ZZ^*$ could be tested at the LHC with future data
- In particular the null hypothesis of Bell inequalities be satisfied, can be excluded using the data of Hi-lumi at the LHC with a significance of 5.6 for $H \rightarrow ZZ^*$ final state
- For WW, ZZ, WZ , CGLMP is violated only in WW and ZZ for invariant masses above 900 GeV and scattering angles close to $\pi/2$. Significance too low at the LHC
- A significance of 2 for WW could be achieved at future muon and FCC colliders and better than 4 for ZZ at the FCC
- We choose suitable observables based on density matrix that can strongly constrain the anomalous Higgs boson HWW and HZZ couplings at the LHC
- We found that limits on these couplings are competitive or even stronger than current experimental bounds based on cross sections

The race is on !

where the Bell inequalities will be first tested at high energies ?

● **Qubit** → tau production at the Z peak at the run2 LHC

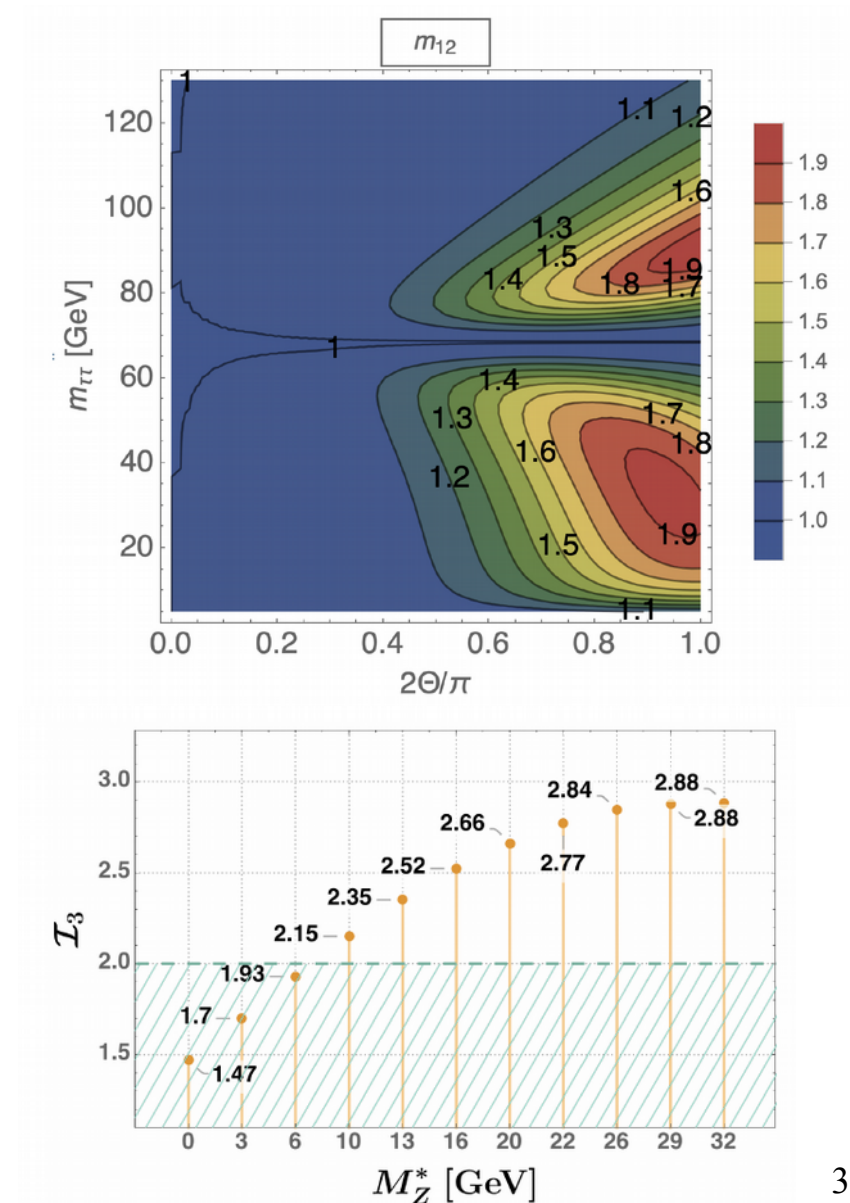
Fabbrichesi, EG, Floreanini, EPJC 83 (2023), 2302.00683 [hep-ph]

● **Qutrit** → Higgs decay $H \rightarrow ZZ^*$ at Hi-Lumi LHC

Aguilar-Saavedra, Bernal, Casas, Moreno, 2209.13441 [hep-ph]

Ashby-Pickering, Barr, Wierzchucka, 2209.13990 [quant-ph]

Fabbrichesi, Floreanini, EG, Marzola, 2302.00683 [hep-ph]



backup slides

Probing entanglement at LHC and future colliders

(recent activity → starting from 2021)

+ top-quark pair production

SM →

- Afik, de Nova, Euro Phys. J Plus 136 (2021) 2003.02280 [quant-ph]
Fabbrichesi, Floreanini, Panizzo, PRL 127 (2021), 2102.11883 [hep-ph]
Severi, Boschi, Maltoni, Sioli, EPJC 82 (2022), 2110.10112 [hep-ph]
Afik, de Nova, Quantum 6 (2022), 2203.05582 [quant-ph]
Aguilar-Saavedra, Casas, EPJC 82 (2022), 2205.00542 [hep-ph]

New Physics →

- Fabbrichesi, EG, Floreanini, EPJC 83 (2023), 2302.00683 [hep-ph]
Aoude, Madge, Maltoni, Mantani, PRD 106 (2022), 2203.05619 [hep-ph]
Severi, Vryonidou, JHEP 01 (2023), 2210.09339 [hep-ph]

+ tau-pair production (Drell-Yan)

- Fabbrichesi, EG, Floreanini, EPJC 83 (2023), 2302.00683 [hep-ph]

+ Λ -hyperons

- Gong, Parida, Tu, Venugopalan, 2107.13007 [hep-ph]

+ Higgs boson → tau pair, two photons

- Fabbrichesi, Gabrielli, Floreanini, EPJC 83 (2023), 2208.11723 [hep-ph]
Altakach, Lambda, Maltoni, Mawatari, Sakurai, 2211.10513 [hep-ph]

→ weak gauge-boson pairs

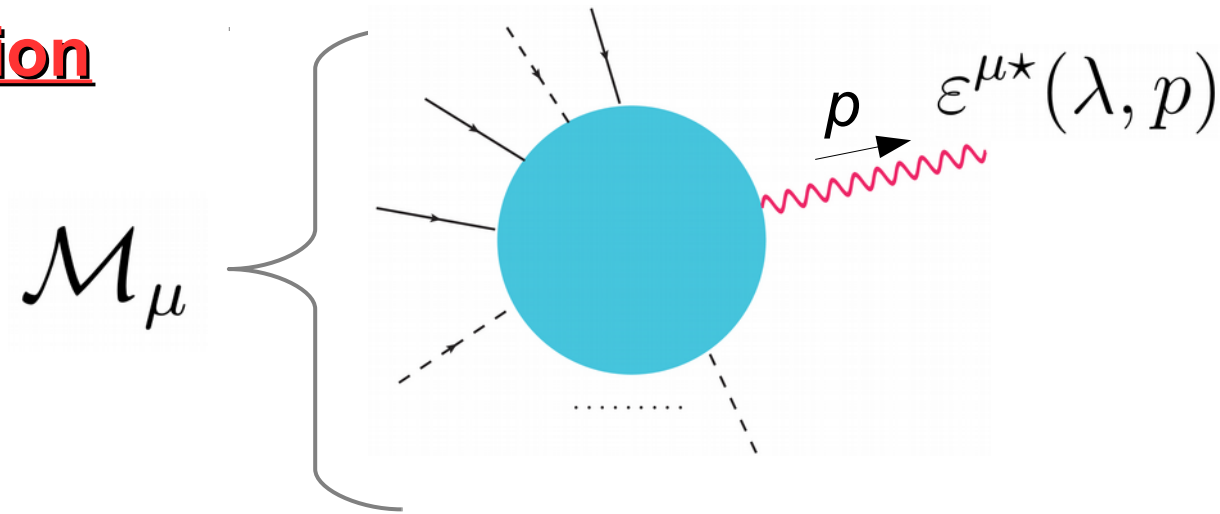
- Alan Barr, PLB 285 (2022), 2106.01377 [hep-ph]
Barr, Caban, Rembielinski, 2204.11063 [hep-ph]
Aguilar-Saavedra, Bernal, Casas, Moreno, 2209.13441 [hep-ph]
Aguilar-Saavedra, 2209.14033 [hep-ph]
Fabbrichesi, Floreanini. EG, Marzola, 2302.00683 [hep-ph]

+ WW, ZZ, WZ (Drell-Yan)

- Ashby-Pickering, Barr, Wierchucka, 2209.13990 [quant-ph]
Fabbrichesi, Floreanini. EG, Marzola, 2302.00683 [hep-ph]

Matrix element for a spin-1 emission

$$\mathcal{M}(\lambda) = \mathcal{M}_\mu \varepsilon^{\mu*}(\lambda, p)$$



Density matrix

$$\rho(\lambda, \lambda') = \frac{\mathcal{M}(\lambda) \mathcal{M}^\dagger(\lambda')}{|\overline{\mathcal{M}}|^2}$$

unpolarized square amplitude

Useful to project density matrix
on the **Gell-Mann** basis

T^a 3x3 Gell-Mann matrices

covariant expression

$$\rho(\lambda, \lambda') = \frac{\mathcal{M}_\mu \mathcal{M}_\nu^\dagger \mathcal{P}^{\mu\nu}_{\lambda\lambda'}(p)}{|\overline{\mathcal{M}}|^2}$$

$$\mathcal{P}^{\mu\nu}_{\lambda\lambda'}(p) = \varepsilon^\mu(p, \lambda)^* \varepsilon^\nu(p, \lambda')$$

$$\rho(\lambda, \lambda') = \left(\frac{1}{3} \mathbb{1} + \sum_{a=1}^8 v^a T^a \right)_{\lambda\lambda'}$$

$$v^a = \frac{1}{2} \text{Tr} [\rho T^a]$$

Spin-1 matrices

$$S_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad S_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad S_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Expressed as a function of **Gell-Mann matrices**

$$S_1 = \frac{1}{\sqrt{2}} (T^1 + T^6), \quad S_2 = \frac{1}{\sqrt{2}} (T^2 + T^7), \quad S_3 = \frac{1}{2} T^3 + \frac{\sqrt{3}}{2} T^8$$

$$\begin{aligned} S_{31} = S_{13} &= \frac{1}{\sqrt{2}} (T^1 - T^6), \\ S_{12} = S_{21} &= T^5, \\ S_{23} = S_{32} &= \frac{1}{\sqrt{2}} (T^2 - T^7) \\ S_{11} &= \frac{1}{2\sqrt{3}} T^8 + T^4 - \frac{1}{2} T^3, \\ S_{22} &= \frac{1}{2\sqrt{3}} T^8 - T^4 - \frac{1}{2} T^3, \\ S_{33} &= T^3 - \frac{1}{\sqrt{3}} T^8, \end{aligned}$$

Gell-Mann basis

$$\begin{aligned} T^1 &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & T^2 &= \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & T^3 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ T^4 &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, & T^5 &= \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, & T^6 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \\ T^7 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, & T^8 &= \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}. \end{aligned}$$

$\mathbb{1}$ being the 3×3 unit matrix

- A suitable observable (**concurrence**) to quantify entanglement in a bipartite (A,B) system for a pure state $|\Psi\rangle$ (with matrix density $\rho = |\Psi\rangle\langle\Psi|$) is defined as

concurrence

$$\mathcal{C}[|\Psi\rangle] = \sqrt{1 - \text{Tr}[(\rho_r)^2]}$$

Rungta, Buzek, Caves, Hillery, Milburn,
PRA 64 (2001)

$r = A$ or B

pure states

vanishes for separable states $|\Psi\rangle = |\Psi_A\rangle \otimes |\Psi_B\rangle$

$$\rho_A = \text{Tr}_B[|\Psi\rangle\langle\Psi|] \text{ and similar for } \rho_B$$

Trace performed in the subsystem B

- For **mixed states** with matrix density $\rho = \sum_i p_i |\Psi_i\rangle\langle\Psi_i|$, $p_i \geq 0$, $\sum_i p_i = 1$

$$\mathcal{C}[\rho] = \inf_{\{|\Psi_i\rangle\}} \sum_i p_i \mathcal{C}[|\Psi_i\rangle]$$

infimum taken over all possible decompositions in pure states

$\mathcal{C}[\rho]$ vanishes for separable states

Bell operator is $\Rightarrow \mathcal{B} =$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{2}{\sqrt{3}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{2}{\sqrt{3}} & 0 & 2 & 0 & 0 \\ 0 & -\frac{2}{\sqrt{3}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{2}{\sqrt{3}} & 0 & 0 & 0 & -\frac{2}{\sqrt{3}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{2}{\sqrt{3}} & 0 \\ 0 & 0 & 2 & 0 & -\frac{2}{\sqrt{3}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{2}{\sqrt{3}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

written on the basis of spin-operators, where

$$S_3 \text{ spin operator} \longrightarrow \text{diag}\{1, 0, -1\}$$

U,V matrices maximizing the Bell observable $\mathcal{I}_3$ in $H \rightarrow WW^*, ZZ^*$

for region $M_W^* = 40$ GeV and $M_Z^* = 32$ GeV

$$U_W = \begin{pmatrix} \frac{4}{11} + \frac{i}{14} & \frac{1}{6} + \frac{9i}{13} & \frac{3}{5} + \frac{i}{14} \\ -\frac{1}{9} - \frac{6i}{7} & 0 & \frac{1}{10} + \frac{i}{2} \\ \frac{4}{11} + \frac{i}{12} & -\frac{1}{7} - \frac{7i}{10} & \frac{3}{5} + \frac{i}{10} \end{pmatrix}, \quad V_W = \begin{pmatrix} -\frac{1}{7} - \frac{7i}{12} & -\frac{7}{10} - \frac{i}{10} & -\frac{1}{9} - \frac{6i}{17} \\ \frac{11}{21} + \frac{i}{17} & 0 & -\frac{6}{7} - \frac{i}{26} \\ -\frac{1}{8} - \frac{3i}{5} & \frac{7}{10} + \frac{i}{8} & -\frac{1}{10} - \frac{5i}{14} \end{pmatrix}$$

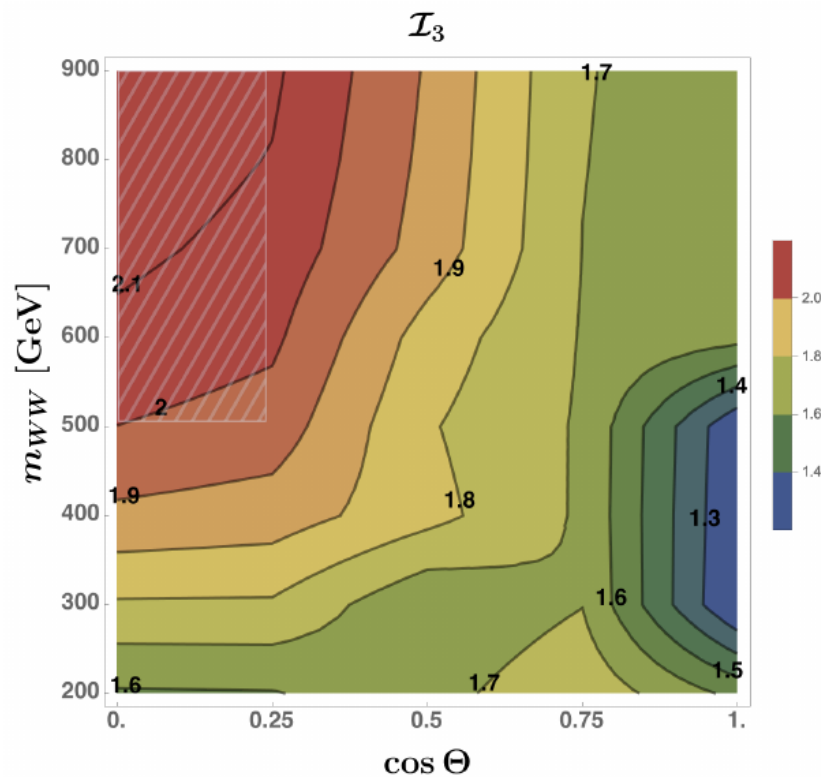
$$U_Z = \begin{pmatrix} -\frac{1}{2} + \frac{3i}{11} & \frac{7}{13} + \frac{5i}{11} & \frac{4}{13} - \frac{3i}{10} \\ -\frac{1}{2} + \frac{3i}{8} & 0 & -\frac{15}{31} + \frac{5i}{8} \\ -\frac{1}{5} + \frac{10i}{19} & -\frac{5}{7} & +\frac{1}{22} - \frac{3i}{7} \end{pmatrix}, \quad V_Z = \begin{pmatrix} -\frac{1}{7} - \frac{5i}{12} & \frac{7}{11} + \frac{2i}{7} & \frac{1}{25} - \frac{5i}{9} \\ \frac{2}{11} + \frac{10i}{13} & 0 & \frac{2}{7} + \frac{6i}{11} \\ \frac{1}{6} + \frac{2i}{5} & -\frac{11}{16} + \frac{i}{5} & -\frac{1}{3} - \frac{4i}{9} \end{pmatrix}$$

approximated matrices within 1% , unitary barring $O(10^{-2})$

U,V matrices maximizing the Bell observable $\mathcal{I}_3$ in $pp \rightarrow WW$

corresponding to the hatched area (see plot below)

$$U_W = \begin{pmatrix} \frac{1}{50} - \frac{5i}{9} & -\frac{1}{6} + \frac{3i}{7} & -\frac{1}{13} + \frac{9i}{13} \\ \frac{1}{4} - \frac{4i}{7} & \frac{2}{9} - \frac{5i}{7} & \frac{1}{5} + \frac{i}{12} \\ \frac{2}{5} - \frac{2i}{5} & -\frac{1}{9} + \frac{4i}{9} & \frac{1}{3} - \frac{3i}{5} \end{pmatrix}, \quad V_W = \begin{pmatrix} -\frac{1}{16} - \frac{4i}{7} & -\frac{2}{11} + \frac{3i}{7} & -\frac{1}{8} + \frac{2i}{3} \\ -\frac{2}{13} + \frac{3i}{5} & -\frac{3}{11} + \frac{5i}{7} & -\frac{1}{5} - \frac{i}{13} \\ \frac{1}{3} - \frac{4i}{9} & -\frac{1}{8} + \frac{3i}{7} & \frac{3}{8} - \frac{3i}{5} \end{pmatrix}$$



approximated matrices within 1% , unitary barring $O(10^{-2})$