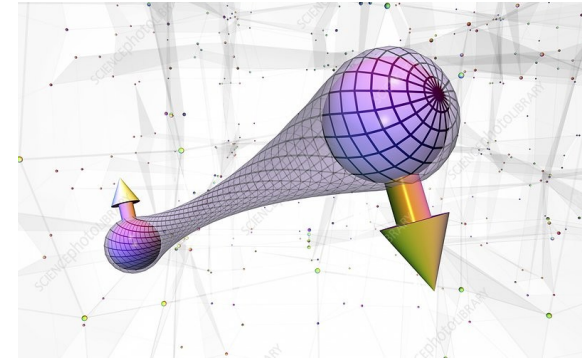


Bell Inequality is violated in $B^0 \rightarrow J/\Psi K^{0*}(892)$ decay

Emidio Gabrielli

University of Trieste, Italy
NICPB, Tallinn, Estonia



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in collaboration with: Marco Fabbrichesi, Roberto Floreanini, and Luca Marzola

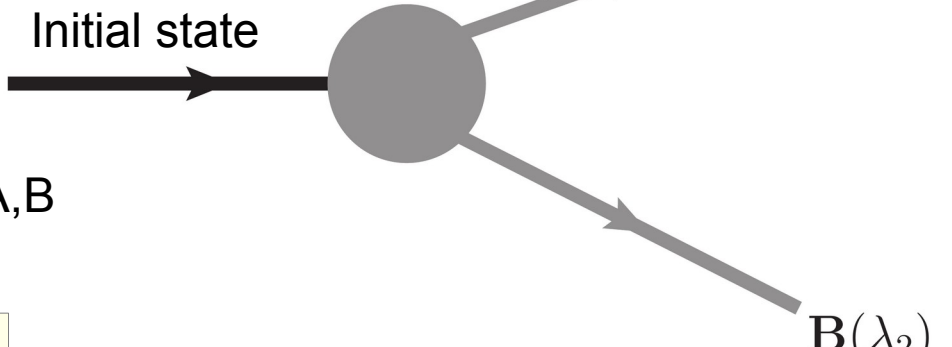
based on: 2305.04982 [hep-ph]

Outlook

- Entanglement as well as Bell inequality expected at high energy
- no test of Bell inequality has been performed at high energy so far
- We focus on **resonant processes** of **scalar/pseudoscalars** decays
- B meson decays in **two spin-1** particles allow to test Bell inequality
- polarization measurements available for many channels (LHCb, Belle)
- we focus on final states of **two spin-1 mesons**: K^* , Φ , J/Ψ , ρ , ..
- use experimental results on polarization amplitudes to compute the correlator of Bell inequality for qutrits
- **From LHCb analysis** → first time observation of **Bell violation** at high energy in B meson decays and in the presence of **WEAK and STRONG** interactions !

Entanglement and Bell inequalities in spin systems

- Requires the knowledge of the **polarization density matrix** of two-particles **A,B** in production
- it can be fully reconstructed from the angular distributions of the single **A,B decay products**
- **but it can also be computed analytically**

Quantum polarized amplitude $\mathcal{M}(\lambda_1, \lambda_2) =$ 

 helicities of A,B

Polarization density matrix ρ

$$\rho(\lambda_1, \lambda'_1, \lambda_2, \lambda'_2) = \frac{\mathcal{M}(\lambda_1, \lambda_2) \mathcal{M}^\dagger(\lambda'_1, \lambda'_2)}{|\overline{\mathcal{M}}|^2}$$

 unpolarized

- knowledge of the full **polarization density matrix** allows to quantify (where possible) entanglement and Bell inequality violations

Quantifying entanglement

in general is a difficult task → simplifies in the case of **pure states**

✓ Pure states (or pure ensembles): the system is composed only by **one** quantum state $|\psi\rangle$

$$\text{Density matrix } \rho = |\psi\rangle\langle\psi| \quad \text{for pure states}$$

✓ in general we do not have pure states → but **mixture of states** $|\psi_n\rangle_{n=1,2,\dots}$

$$\rho = \sum_n p_n |\psi_n\rangle\langle\psi_n|, \quad \sum_n p_n = 1 \quad \Rightarrow \quad \text{Tr}[\rho] = 1$$

$\langle A \rangle = \text{Tr}[\rho A]$ → expectation value of operators (A) → can be computed in any base since Trace is base independent

✓ For pure states

$$\rho \doteq \begin{pmatrix} 0 & & & & & & 0 \\ & 0 & & & & & \\ & & \ddots & & & & \\ & & & 0 & & & \\ & & & & 1 & & \\ & & & & & 0 & \\ & & & & & & 0 \\ & & & & & & & \ddots & \\ 0 & & & & & & & & 0 \end{pmatrix} \quad (\text{diagonal form})$$

$$\rho^2 = \rho$$

$$\text{Tr}[\rho^2] = 1$$

quantifying entanglement in pure states

2302.00683 [hep-ph]

- If the bipartite (A,B) system is a pure state it is possible to quantify its entanglement via

Entropy of entanglement

$$\mathcal{E}[\rho] = -\text{Tr} [\rho_A \log \rho_A] = -\text{Tr} [\rho_B \log \rho_B]$$

density matrix of reduced A or B system

equality holds if and only if the bipartite is separable

$$0 \leq \mathcal{E}[\rho] \leq \ln 3$$

(Von Neumann Entropy)

↓
separable states

↓
Maximum entanglement

- Very useful in the case of bipartite (A,B) system of qutrit for which there is not a solution for the general case

Two-QUTRITS systems

consider two massive spin-1 particles

Useful to decompose **density matrix** on the basis of tensor products of **Gell-Mann matrices** T^a

9x9 matrix

Unpolarized

$$[A \otimes B]_{ii'jj'} = A_{ii'} B_{jj'}$$

$$\rho(\lambda_1, \lambda'_1, \lambda_2, \lambda'_2) = \left(\frac{1}{9} [\mathbb{1} \otimes \mathbb{1}] + \sum_a f_a [\mathbb{1} \otimes T^a] + \sum_a g_a [T^a \otimes \mathbb{1}] + \sum_{ab} h_{ab} [T^a \otimes T^b] \right)_{\lambda_1 \lambda'_1, \lambda_2 \lambda'_2}$$

helicities ↓

$$f_a = \frac{1}{6} \text{Tr} [\rho (\mathbb{1} \otimes T^a)]$$

Spin polarization of particle 1

$$g_a = \frac{1}{6} \text{Tr} [\rho (T^a \otimes \mathbb{1})]$$

Spin polarization of particle 2

$$h_{ab} = \frac{1}{4} \text{Tr} [\rho (T^a \otimes T^b)]$$

Spin **correlations** of particles 1 and 2

● we can also **extract them from data**, using the angular distributions of their decay products

Bell inequalities for Two-QUTRITS

- take a bi-partite system with components A,B
- perform two measurements $\longrightarrow (A_1, A_2) \quad (B_1, B_2)$ each can take values $\rightarrow \{0, 1, 2\}$ on A,B
- consider the following correlator $\mathcal{I}_3$ for probability measurements

$$\begin{aligned} \mathcal{I}_3 = & P(A_1 = B_1) + P(B_1 = A_2 + 1) + P(A_2 = B_2) + P(B_2 = A_1) \\ & - P(A_1 = B_1 - 1) - P(A_1 = B_2) - P(A_2 = B_2 - 1) - P(B_2 = A_1 - 1) \end{aligned}$$

$\mathcal{I}_3$ can be expressed as

$$\mathcal{I}_3 = \text{Tr}[\rho \mathcal{B}]$$

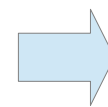
with $\mathcal{B}$ a suitable Bell operator

Collins, Gisin, Linden,
Massar, Popescu, PRL 88 (2002)

Bell inequalities for two-qutrits

For deterministic local models

$$\mathcal{I}_3 \leq 2$$



QM for qutrits can violate this
inequality with upper bound = 4

- problem of finding optimal choice has been found

for the maximal entangled state

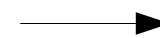
$$|\Psi_+\rangle = \frac{1}{\sqrt{3}} \sum_{i=1}^3 |i\rangle \otimes |i\rangle$$

$$\rho = |\Psi_+\rangle \langle \Psi_+|$$

given by the Bell operator $\mathcal{B}$ (see backup slides)

- Still freedom from Bob and Alice to choose directions where to measure the spin in such a way to maximize $\mathcal{I}_3$
- this can be achieved by performing unitary transformations on the Bell operator as

$$\mathcal{B} \rightarrow (U \otimes V)^\dagger \cdot \mathcal{B} \cdot (U \otimes V)$$



U, V are unitary 3x3 matrices
(depend on the kinematic of the process)

scalar/pseudoscalar decay in two vectors $S \rightarrow V_1 V_2$

- $V_1 V_2$ comes out as a pure state

Fabbrichesi, Floreanini, EG, Marzola
2304.02403 [hep-ph]
2305.04982 [hep-ph]

Approach based on helicity amplitudes

helicity amplitude $\Rightarrow$

$$h_\lambda = \langle V_1(\lambda) V_2(-\lambda) | \mathcal{H} | B \rangle \quad \text{with} \quad \lambda = (+, 0, -)$$

generic scalar meson (for instance B meson)

From angular momentum conservation
there are only 3 helicity amplitude combination

helicities defined with respect to
z-direction (spin quantization axis)
in rest frame of one of the two Vectors

$$|\Psi\rangle = \frac{1}{\sqrt{|H|^2}} \left[h_+ |V_1(+) V_2(-)\rangle + h_0 |V_1(0) V_2(0)\rangle + h_- |V_1(-) V_2(+)\rangle \right]$$

$$|H|^2 = |h_0|^2 + |h_+|^2 + |h_-|^2$$

Relative weight of $|0\ 0\rangle$ with respect to $|+\ -\rangle$ or $|-\ +\rangle$ controlled by angular momentum conservation

Density matrix for a pure $V_1 V_2$ state can be written as

$$\rho = |\Psi\rangle\langle\Psi|$$

Fabbrichesi, Floreanini, EG, Marzola
2304.02403 [hep-ph]
2305.04982 [hep-ph]

Master formula

in helicity basis

$$\rho = \frac{1}{|H|^2} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & h_+ h_+^* & 0 & h_+ h_0^* & 0 & h_+ h_-^* & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & h_0 h_+^* & 0 & h_0 h_0^* & 0 & h_0 h_-^* & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & h_- h_+^* & 0 & h_- h_0^* & 0 & h_- h_-^* & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Proved also using most general Lorentz invariant structure for amplitude decay of $S \rightarrow V_1 V_2$

$$\frac{h_0}{|H|} = A_0, \quad \frac{h_+}{|H|} = \frac{A_{\parallel} + A_{\perp}}{\sqrt{2}} \quad \text{and} \quad \frac{h_-}{|H|} = \frac{A_{\parallel} - A_{\perp}}{\sqrt{2}}$$

Polarization amplitudes directly measured in experimental analysis of B meson decays !

- One can also compute the polarization density matrix from the most general Lorentz invariant structure of the amplitude $\mathcal{M}$ for $\mathbf{S} \rightarrow \mathbf{V}_1 \mathbf{V}_2$

$$\mathcal{M}(\lambda_1, \lambda_2) = M_{\mu\nu} \varepsilon^{\mu*}(k_1, \lambda_1) \varepsilon^{\nu*}(k_2, \lambda_2)$$

$$M^{\mu\nu} = A_1 g^{\mu\nu} + A_2 k_1^\nu k_2^\mu + A_3 \varepsilon^{\mu\nu\alpha\beta} k_{1\alpha} k_{2\beta}$$

$$\rho = \frac{\mathcal{M}_{\mu\nu} \mathcal{M}_{\mu'\nu'}^\dagger}{|\overline{\mathcal{M}}|^2} \left[\mathcal{P}^{\mu\mu'}(k_1) \otimes \mathcal{P}^{\nu\nu'}(k_2) \right] \quad \text{omitting helicity indices}$$

where $\mathcal{P}_{\lambda\lambda'}^{\mu\nu}(p)$ is the covariant projector (see backup slides for details)

$$\mathcal{P}_{\lambda\lambda'}^{\mu\nu}(p) = \frac{1}{3} \left(-g^{\mu\nu} + \frac{p^\mu p^\nu}{M_V^2} \right) \delta_{\lambda\lambda'} - \frac{i}{2M_V} \varepsilon^{\mu\nu\alpha\beta} p_\alpha n_\beta^i (S_i)_{\lambda\lambda'} - \frac{1}{2} n_i^\mu n_j^\nu (S_{ij})_{\lambda\lambda'}$$

right-handed basis for spin

Spin-1 matrices

$$S_{ij} = S_i S_j + S_j S_i - \frac{4}{3} \mathbb{1} \delta_{ij} \quad i \in \{1, 2, 3\}$$

- using the relation between the spin matrices $\mathbf{S}_i$, $\mathbf{S}_{ij}$ and the Gell-Mann matrices $\mathbf{T}^a$,

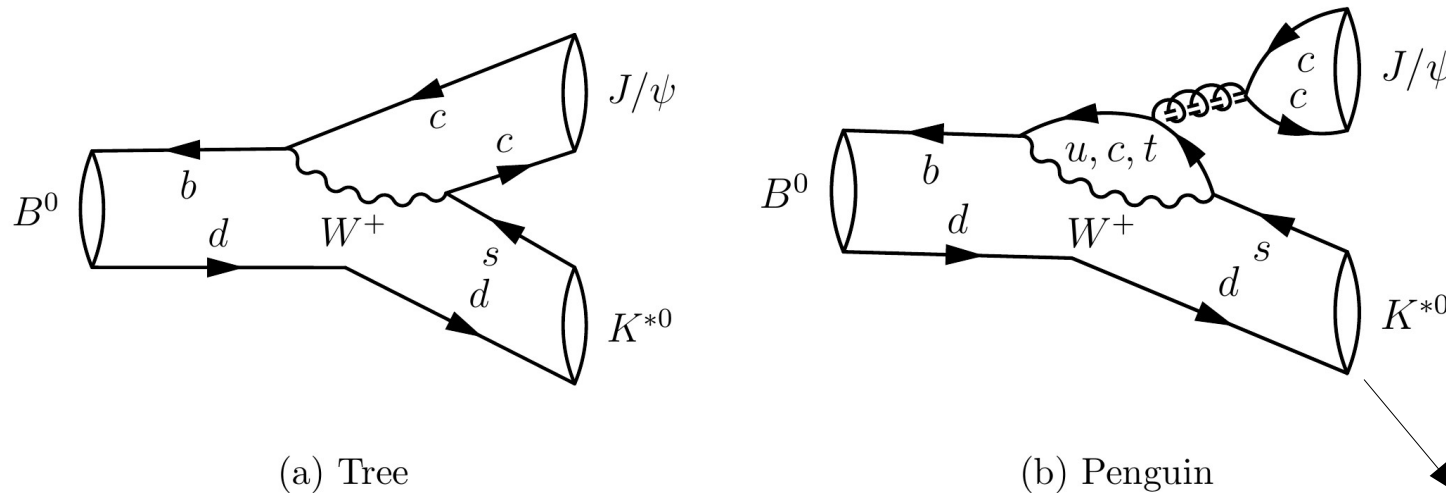
one can extract the polarization coefficients f_a, g_a, h_{ab} in the Gell-Mann basis

Non-vanishing matrix elements of the density matrix for $\mathbf{S} \rightarrow \mathbf{V}_1 \mathbf{V}_2$
in the **basis of helicity amplitudes** and in the **Gell-Mann basis**



$$\begin{aligned}\hat{h}_- \hat{h}_-^* &= \frac{1}{9} \left[1 + 3\sqrt{3} (f_8 - 2g_8 - 2h_{38}) + 9f_3 - 6h_{88} \right], \\ \hat{h}_0 \hat{h}_-^* &= h_{16} + i (h_{17} - h_{26}) + h_{27}, \\ \hat{h}_+ \hat{h}_-^* &= h_{44} + i (h_{45} - h_{54}) + h_{55}, \\ \hat{h}_0 \hat{h}_0^* &= \frac{1}{9} \left[1 - 9(f_3 + g_3 - h_{33}) + 3\sqrt{3} (f_8 + g_8 - h_{38} - h_{83}) + 3h_{88} \right], \\ \hat{h}_+ \hat{h}_0^* &= h_{61} + i (h_{62} - h_{71}) + h_{72}, \\ \hat{h}_+ \hat{h}_+^* &= \frac{1}{9} \left[1 + 3\sqrt{3} (g_8 - 2f_8 - 2h_{83}) + 9g_3 - 6h_{88} \right],\end{aligned}$$

Bell inequality violation in B meson decay $B \rightarrow J/\psi K^*$



Smaller contribution, not relevant for Bell inequality, but crucial for FCNC, CP violation tests

Measured by several collaborations, most recent one

R.Aaij et al [LHCb], PRD, 88 (2013) 052002, 1307.2782 [hep-exp]

From each decay

$$J/\psi \rightarrow \mu^+ \mu^-$$

$$K^* \rightarrow K^+ \pi^-$$

➡ We can measure polarization amplitudes

- from polarization amplitudes we can reconstruct the density matrix and compute the $\mathcal{I}_3$ observable for qutrits

- 4 independent parameters: 3 amplitudes modulus and 2 relative phases

Polarization amplitudes

$$A_{\parallel} = |A_{\parallel}| e^{i\delta_{\parallel}}$$

$$A_{\perp} = |A_{\perp}| e^{i\delta_{\perp}}$$

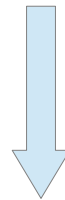
$$A_0 = |A_0| e^{i\delta_0}$$

We choose $\delta_0 = 0$

Phases due to final state interactions

induced by strong interactions due to rescattering

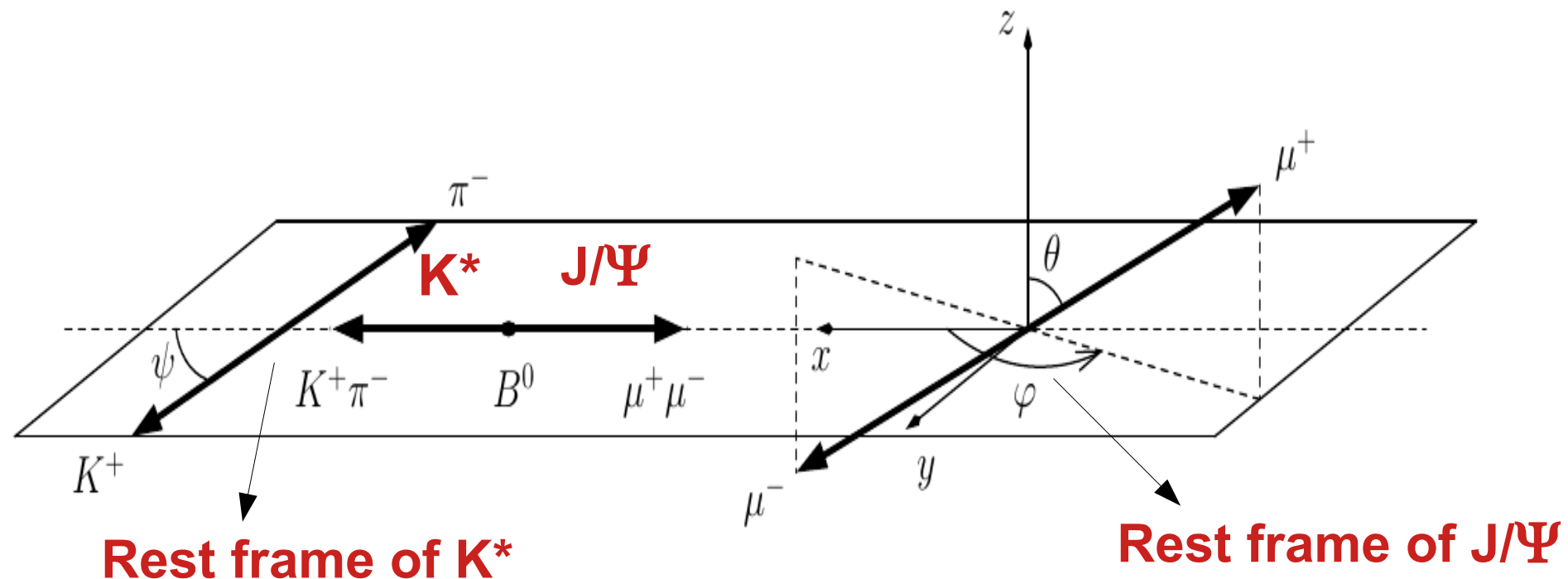
Distance when both have decayed
can be estimated to be $\sim 1100 \text{ fm}$



Decays take place outside the range of strong interaction at the time of their production ($3 \times 10^{-5} \text{ fm}$ gluon exchange) as well as of final state interactions (1.5 fm exchange of pions)

From angular distribution one can extract the polarization amplitudes and phases by a fit of coefficients h_k (see next slide)

$$\frac{d^4\Gamma(B^0 \rightarrow J/\psi K^{*0})}{dt d\Omega} \propto e^{-\Gamma_d t} \sum_{k=1}^{10} h_k f_k(\Omega) \quad \Omega = \{\cos \theta, \cos \psi, \varphi\}$$



FIT of coefficients h_k

$$\Omega = \{\cos \theta, \cos \psi, \varphi\}$$

1307.2782 [hep-exp]

Dighe, Dunietz, Lipkin,
Rosner, PLB 369 144
(1996), **9511363 [hep-ph]**

k	h_k	$f_k(\Omega)$
1	$ A_0 ^2$	$\frac{9}{32\pi} 2 \cos^2 \psi (1 - \sin^2 \theta \cos^2 \varphi)$
2	$ A_{\parallel} ^2$	$\frac{9}{32\pi} \sin^2 \psi (1 - \sin^2 \theta \sin^2 \varphi)$
3	$ A_{\perp} ^2$	$\frac{9}{32\pi} \sin^2 \psi \sin^2 \theta$
4	$ A_{\parallel} A_{\perp} \sin(\delta_{\perp} - \delta_{\parallel})$	$-\frac{9}{32\pi} \sin^2 \psi \sin 2\theta \sin \varphi$
5	$ A_0 A_{\parallel} \cos(\delta_{\parallel})$	$\frac{9}{32\pi\sqrt{2}} \sin 2\psi \sin^2 \theta \sin 2\varphi$
6	$ A_0 A_{\perp} \sin(\delta_{\perp})$	$\frac{9}{32\pi\sqrt{2}} \sin 2\psi \sin 2\theta \cos \varphi$
7	$ A_S ^2$	$\frac{3}{32\pi} 2(1 - \sin^2 \theta \cos^2 \varphi)$
8	$ A_{\parallel} A_S \cos(\delta_{\parallel} - \delta_S)$	$\frac{3}{32\pi} \sqrt{6} \sin \psi \sin^2 \theta \sin 2\varphi$
9	$ A_{\perp} A_S \sin(\delta_{\perp} - \delta_S)$	$\frac{3}{32\pi} \sqrt{6} \sin \psi \sin 2\theta \cos \varphi$
10	$ A_0 A_S \cos(\delta_S)$	$\frac{3}{32\pi} 4\sqrt{3} \cos \psi (1 - \sin^2 \theta \cos^2 \varphi)$

(A_S contribution from
non-resonant J/Y K^* amplitude)

$$|A_{\parallel}|^2 = 0.227 \pm 0.004 \text{ (stat)} \pm 0.011 \text{ (sys)}$$
$$|A_{\perp}|^2 = 0.201 \pm 0.004 \text{ (stat)} \pm 0.008 \text{ (sys)}$$
$$\delta_{\parallel} \text{ [rad]} = -2.94 \pm 0.02 \text{ (stat)} \pm 0.03 \text{ (sys)}$$
$$\delta_{\perp} \text{ [rad]} = 2.94 \pm 0.02 \text{ (stat)} \pm 0.02 \text{ (sys)},$$

+ error matrix correlation (included)

$$|A_0|^2 + |A_{\perp}|^2 + |A_{\parallel}|^2 = 1$$

$$\mathcal{I}_3 = \text{Tr}[\rho \mathcal{B}]$$

maximized using $\mathcal{B} \rightarrow (U \otimes V)^{\dagger} \cdot \mathcal{B} \cdot (U \otimes V)$

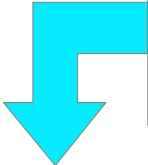
(see backup slides for U,V)

classical locality requires $\mathcal{I}_3 < 2$

Bell inequality →

$$\mathcal{I}_3 = 2.548 \pm 0.015$$

→ $B^0 \rightarrow J/\psi K^*(892)^0$


$$\mathcal{I}_3 < 2 \text{ is violated with a significance of } 36\sigma$$

First time violation of Bell inequalities in qutrits and in the presence of weak and strong interactions

Quantum entanglement and Bell violation in other B decays

2305.04982 [hep-ph]

$0 < \text{von Neumann entropy} < \ln[3] \sim 1.1$

$$\mathcal{I}_3 = \text{Tr}[\rho \mathcal{B}]$$

	$\mathcal{E}$	$\mathcal{I}_3$	
• $B^0 \rightarrow J/\psi K^*(892)^0$ [5]	0.756 ± 0.009	2.548 ± 0.015	
• $B^0 \rightarrow \phi K^*(892)^0$ [20]	$0.707 \pm 0.133^*$	$2.417 \pm 0.368^*$	$\rightarrow 1.1\sigma$
• $B^0 \rightarrow \rho K^*(892)^0$ [21]	$0.450 \pm 0.067^*$	$2.208 \pm 0.129^*$	$\rightarrow 1.6\sigma$
• $B_s \rightarrow \phi\phi$ [22]	$0.734 \pm 0.050^*$	$2.525 \pm 0.084^*$	$\rightarrow 6.2\sigma$
• $B_s \rightarrow J/\psi\phi$ [23]	0.731 ± 0.032	2.462 ± 0.080	$\rightarrow 5.8\sigma$

● all these decays show a high level of entanglement

● values with * refer to lower bound on errors (because correlation matrix is not available)

significance on Bell inequality should increase after including error correlations

- [20] K. F. Chen *et al.* [Belle], Phys. Rev. Lett. **94**, 221804 (2005) [arXiv:hep-ex/0503013 [hep-ex]].
- [21] R. Aaij *et al.* [LHCb], JHEP **05**, 026 (2019) [arXiv:1812.07008 [hep-ex]].
- [22] R. Aaij *et al.* [LHCb], [arXiv:2304.06198 [hep-ex]].
- [23] G. Aad *et al.* [ATLAS], Eur. Phys. J. C **81**, no.4, 342 (2021) [arXiv:2001.07115 [hep-ex]].
- [5] R. Aaij *et al.* [LHCb], Phys. Rev. D **88**, 052002 (2013) [arXiv:1307.2782 [hep-ex]].

Closing the **locality loophole** in $B \rightarrow \Phi\Phi$

- One must consider the case where the decay products are the same particles (same width)
- $B \rightarrow \Phi\Phi$ decay offers this possibility

events satisfying the space-like condition



$$\frac{|t_1 - t_2| c}{(t_1 + t_2) v} < 1$$

$t_{1,2}$ → time of decays

Using the fact that decay times follow the PDF distribution in time (t) → $P(t) \sim \exp[-t \Gamma / \gamma]$

Γ → the particle width at rest
 γ → the Lorentz factor

- we find that almost **90% of events** satisfies this condition
- Locality loophole can be closed

Summary

- Bell inequality is violated in B meson decay $B \rightarrow J/\psi K^*$
- First time observation (with 36σ level significance) in Weak and Strong interactions and at high energies ($\sim \text{GeV}$) and in quark systems
- Observed also in other B mesons decays (with less significance)
- $B \rightarrow \phi \phi$ violates Bell inequality at 6σ level and it is also locality loophole free
→ higher significance could be achieved if error correlations are included
- Final remark: LHCb experiment was not designed to test the Bell inequalities and quantum entanglement

Backup slides

- Bell operator on the basis of S_3 spin operator

$$\mathcal{B} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{2}{\sqrt{3}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{2}{\sqrt{3}} & 0 & 2 & 0 & 0 \\ 0 & -\frac{2}{\sqrt{3}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{2}{\sqrt{3}} & 0 & 0 & 0 & -\frac{2}{\sqrt{3}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{2}{\sqrt{3}} & 0 \\ 0 & 0 & 2 & 0 & -\frac{2}{\sqrt{3}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{2}{\sqrt{3}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

- Maximization of $\mathbf{I}_3$ achieved by $\mathcal{B} \rightarrow (U \otimes V)^\dagger \cdot \mathcal{B} \cdot (U \otimes V)$

- approximated matrices up to 1% for the central value of $\mathbf{I}_3$ for $B^0 \rightarrow J/\psi K^*$

$$U = \begin{pmatrix} \frac{5}{14} - \frac{8i}{13} & \frac{1}{20} - \frac{4i}{7} & \frac{1}{4} - \frac{i}{3} \\ 0 & \frac{1}{18} - \frac{7i}{12} & -\frac{7}{15} + \frac{11i}{17} \\ -\frac{4}{11} + \frac{3i}{5} & \frac{1}{17} - \frac{4i}{7} & \frac{1}{4} - \frac{i}{13} \end{pmatrix}, \quad V = \begin{pmatrix} -\frac{5}{12} - \frac{i}{99} & -\frac{2}{9} + \frac{8i}{15} & \frac{1}{51} + \frac{5i}{7} \\ -\frac{4}{5} - \frac{i}{38} & \frac{3}{13} - \frac{6i}{11} & 0 \\ -\frac{5}{12} - \frac{i}{59} & -\frac{3}{13} + \frac{9i}{17} & -\frac{5i}{7} \end{pmatrix}$$

covariant polarization vector of spin-1

rest frame limit

$$\varepsilon^\mu(p, \lambda) \xrightarrow{(\beta \rightarrow 0)} \psi_\pm, \psi_0$$

helicity $\lambda = \pm 1, 0$

β is the velocity

boosted base

$$n_1^\mu = (0, \hat{\mathbf{n}}), \quad n_2^\mu = (0, \hat{\mathbf{r}}), \quad n_3^\mu = \frac{E}{M}(\beta, \hat{\mathbf{k}})$$

Covariant Projector

$$\mathcal{P}_{\lambda\lambda'}^{\mu\nu}(p) = \varepsilon^\mu(p, \lambda)^* \varepsilon^\nu(p, \lambda')$$

master formula

$$\varepsilon^{0123} = 1$$

$$= \frac{1}{3} \left(-g^{\mu\nu} + \frac{p^\mu p^\nu}{M^2} \right) \delta_{\lambda\lambda'} - \frac{i}{2M} \epsilon^{\mu\nu\alpha\beta} p_\alpha n_\beta^i (S_i)_{\lambda\lambda'} - \frac{1}{2} n_i^\mu n_j^\nu (S_{ij})_{\lambda\lambda'}$$

$S_i, i \in \{1, 2, 3\} \rightarrow$ rotation matrices for spin-1 particle

H.S. Song, *Lett. Nuovo Cim.* 25 (1979)
S.Y. Choi, T. Lee, H.S. Song, *PRD* 40 (1989)
Fabbrichesi, Floreanini, EG, Marzola,
2302.00683 [hep-ph]

$$S_{ij} = S_i S_j + S_j S_i - \frac{4}{3} \mathbb{1} \delta_{ij} \quad (\text{see backup slides})$$

basis correspondence for $(S_i)_{\lambda\lambda'}$ $|+\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad |0\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad |-\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ corresponding to eigenevalues $\lambda = \pm 1, 0$

Spin-1 matrices

$$S_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad S_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad S_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Expressed as a function of **Gell-Mann matrices**

$$S_1 = \frac{1}{\sqrt{2}} (T^1 + T^6), \quad S_2 = \frac{1}{\sqrt{2}} (T^2 + T^7), \quad S_3 = \frac{1}{2} T^3 + \frac{\sqrt{3}}{2} T^8$$

$$S_{31} = S_{13} = \frac{1}{\sqrt{2}} (T^1 - T^6),$$

$$S_{12} = S_{21} = T^5,$$

$$S_{23} = S_{32} = \frac{1}{\sqrt{2}} (T^2 - T^7)$$

$$S_{11} = \frac{1}{2\sqrt{3}} T^8 + T^4 - \frac{1}{2} T^3,$$

$$S_{22} = \frac{1}{2\sqrt{3}} T^8 - T^4 - \frac{1}{2} T^3,$$

$$S_{33} = T^3 - \frac{1}{\sqrt{3}} T^8,$$

Gell-Mann basis

$$\begin{aligned} T^1 &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & T^2 &= \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & T^3 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ T^4 &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, & T^5 &= \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, & T^6 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \\ T^7 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, & T^8 &= \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}. \end{aligned}$$